

## Accurate Expressions for the Dewpoint and Frost Point Derived from the Rankine–Kirchhoff Approximations

DAVID M. ROMPS<sup>a,b</sup>

<sup>a</sup> *Department of Earth and Planetary Science, University of California, Berkeley, Berkeley, California*

<sup>b</sup> *Climate and Ecosystem Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, California*

(Manuscript received 1 October 2020, in final form 10 March 2021)

**ABSTRACT:** Accurate, explicit, and analytic expressions are derived for the dewpoint and frost point as functions of temperature and relative humidity. These are derived theoretically in terms of physical constants using the Rankine–Kirchhoff approximations, which assume an ideal gas, fixed heat capacities, and zero specific volume of condensates. Compared to modern laboratory measurements, the expressions are accurate to within a few hundredths of a degree over the full range of Earth-relevant temperatures, from 180 to 273 K for the frost point and 230 to 330 K for the dewpoint.

**KEYWORDS:** Humidity; Thermodynamics; Water vapor

John Dalton first introduced a method for reliably measuring the dewpoint, accomplished by determining the highest temperature of water in a glass container that condenses water on its exterior (Dalton 1802b,a). This temperature was originally referred to as the “vapor point” or “point of condensation,” but the term “dewpoint” came into use shortly thereafter (Howard 1818; Dalton 1824). In the years since, many different analytic expressions have been proposed for the dewpoint in terms of temperature and relative humidity (e.g., Mitchell 1967; Wanielista et al. 1997; Sargent 1980; Urbank et al. 2001; Lawrence 2005), but only one has been derived theoretically and that one, as we will see, has errors that can exceed 1 K. This short note derives, from first principles, analytic expressions for the dewpoint and frost point that are accurate to within hundredths of a degree.

The dewpoint  $T_d$  is the temperature to which air must be cooled at constant pressure to reach saturation with respect to liquid water. At the dewpoint, the vapor pressure  $p_v$  equals the saturation vapor pressure with respect to liquid  $p_v^{*l}(T_d)$ ; here, the asterisk denotes saturation and the superscript  $l$  reminds us that this is with respect to a planar surface of liquid. In the absence of phase changes, the partial pressure of water vapor does not change as the air is cooled isobarically, so we can write  $p_v$  as  $\text{RH}_l p_v^{*l}(T)$ , where  $T$  is the original temperature of the air and  $\text{RH}_l$  is the original relative humidity of the air with respect to liquid water. Therefore,  $p_v^*(T_d) = \text{RH}_l p_v^{*l}(T)$ , which can be written as (e.g., Bosen 1958)

$$\text{RH}_l = \frac{p_v^{*l}(T_d)}{p_v^{*l}(T)}. \tag{1}$$

Given an expression for the function  $p_v^{*l}$ , Eq. (1) gives  $\text{RH}_l$  as an explicit function of temperature  $T$  and dewpoint  $T_d$ . Likewise, given the frost point  $T_f$ , we can calculate the relative humidity with respect to solid water  $\text{RH}_s$ , as

$$\text{RH}_s = \frac{p_v^{*s}(T_f)}{p_v^{*s}(T)}, \tag{2}$$

where  $p_v^{*s}$  is the saturation vapor pressure with respect to a planar surface of solid water, i.e., ice.

Given  $T$  and  $\text{RH}_l$  (or  $\text{RH}_s$ ), the most accurate value of the dew (or frost) point is obtained by using an empirical expression for  $p_v^{*l}$  (or  $p_v^{*s}$ ) that has been obtained by fitting to laboratory measurements such as Eq. (10) of Murphy and Koop (2005) [or Eq. (4) of Wagner et al. 2011] and then solving for  $T_d$  (or  $T_f$ ) using a numerical root solver. Our goal here, however, is to derive expressions for  $T_d$  and  $T_f$  that are not only accurate, but also explicit and analytic in terms of temperature and relative humidity. Therefore, we must invert  $p_v^{*l}$ . For this purpose, it is common to use the empirical Magnus expression for  $p_v^{*l}$  (August 1828; Magnus 1844), which can be inverted analytically (Mitchell 1967; Lawrence 2005). But we have here one other objective, which is to derive an expression for the dewpoint that is not only accurate, explicit, and analytic, but also derived theoretically.

To this end, we will use the approximations of Rankine (1866) and Kirchhoff (1858), which assume an ideal gas, constant heat capacities, and zero specific volume of condensates. Following Romps (2021, manuscript submitted to *Quart. J. Roy. Meteor. Soc.*), we may refer to this trio of assumptions as the Rankine–Kirchhoff (RK) approximations, which have been used to derive accurate, explicit, and analytic expressions for the equivalent potential temperature (Romps and Kuang 2010; Hauf and Höller 1987; Emanuel 1994), the quantity conserved by an adiabatically lifted parcel (Romps 2015), and the lifting condensation level (Romps 2017). Here, we aim to add to that list the dewpoint and frost point.

Using the Rankine–Kirchhoff approximations, we can arrive at the Rankine–Kirchhoff equations for the saturation vapor pressures (Rankine 1866; Kirchhoff 1858)

$$p_v^{*l} = p_{\text{trip}} \left( \frac{T}{T_{\text{trip}}} \right)^{(c_{pv} - c_{vl})/R_v} \exp \left[ \frac{E_{0v} - (c_{vv} - c_{vl})T_{\text{trip}}}{R_v} \right] \times \left( \frac{1}{T_{\text{trip}}} - \frac{1}{T} \right), \tag{3}$$

---

Corresponding author: David M. Romps, romps@berkeley.edu

$$p_v^{*,s} = p_{\text{trip}} \left( \frac{T}{T_{\text{trip}}} \right)^{(c_{pv} - c_{vs})/R_v} \exp \left[ \frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\text{trip}}}{R_v} \right] \times \left( \frac{1}{T_{\text{trip}}} - \frac{1}{T} \right), \quad (4)$$

where  $R_v$  is the specific gas constant for water vapor;  $c_{vv}$ ,  $c_{vl}$ ,  $c_{vs}$  are the specific heat capacities at constant volume of water vapor, liquid water, and solid water, respectively;  $c_{pv} = c_{vv} + R_v$  is the specific heat capacity of water vapor at constant pressure;  $p_{\text{trip}}$  and  $T_{\text{trip}}$  are the triple-point vapor pressure and temperature, respectively;  $E_{0v}$  is the difference in specific internal energy between water vapor and liquid water at the triple point; and  $E_{0s}$  is the difference in specific internal energy between liquid water and solid water at the triple point. Following [Romps \(2017\)](#), a set of optimized values for these constants are  $c_{vv} = 1418 \text{ J kg}^{-1} \text{ K}^{-1}$ ,  $p_{\text{trip}} = 611.65 \text{ Pa}$ ,  $T_{\text{trip}} = 273.16 \text{ K}$ ,  $E_{0v} = 2.3740 \text{ MJ kg}^{-1}$ ,  $E_{0s} = 0.3337 \text{ MJ kg}^{-1}$ ,  $R_v = 461 \text{ J kg}^{-1} \text{ K}^{-1}$ ,  $c_{vl} = 4119 \text{ J kg}^{-1} \text{ K}^{-1}$ , and  $c_{vs} = 1861 \text{ J kg}^{-1} \text{ K}^{-1}$ .

Using Eq. (3), Eq. (1) can be inverted to give an explicit and analytic expression for the dewpoint  $T_d$ ,

$$T_d = c \left[ W_{-1} \left( \text{RH}_l^{R_v/(c_{vl} - c_{pv})} c e^c \right) \right]^{-1} T, \quad (5)$$

$$c \equiv \frac{E_{0v} - (c_{vv} - c_{vl})T_{\text{trip}}}{(c_{pv} - c_{vl})T}, \quad (6)$$

where  $W_{-1}$  is the lower branch of the Lambert  $W$  function. Likewise, using Eq. (4), Eq. (2) can be inverted to give the frost point  $T_f$ ,

$$T_f = c \left[ W_0 \left( \text{RH}_s^{R_v/(c_{vs} - c_{pv})} c e^c \right) \right]^{-1} T, \quad (7)$$

$$c \equiv \frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\text{trip}}}{(c_{pv} - c_{vs})T}, \quad (8)$$

where  $W_0$  is the principal branch of the Lambert  $W$  function. The reason the dewpoint and frost point use different branches is that  $c_{vl} - c_{pv}$  and  $c_{vs} - c_{pv}$  have different signs for the choice of heat capacities used here. If different heat capacities were used for which  $c_{vs} > c_{pv}$ , then the lower branch would be used for the frost point instead.

In the case of the frost point, the value of  $e^c$  can be very large and can cause the argument of  $W_0$  to exceed the largest number that can be represented using the IEEE double-precision floating-point format, which is about  $1.8 \times 10^{308}$ . If the logarithm of the argument exceeds  $709 \approx \log(10^{308})$ , then we can use an approximation for  $W_0$ ,

$$W_0(x) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2 + L_2)}{2L_1^2} + \frac{L_2(6 - 9L_2 + 2L_2^2)}{6L_1^3} + \frac{L_2(-12 + 36L_2 - 22L_2^2 + 3L_2^3)}{12L_1^4},$$

where  $L_1 = \log(x)$  and  $L_2 = \log(L_1)$  ([Corless et al. 1996](#)). When  $L_1$  exceeds 709, this approximation replicates  $W_0$  with a relative error that is smaller than double-precision rounding error.

Equations (7) and (8) give the frost point in terms of  $\text{RH}_s$ , but relative humidity is most commonly stated with respect to

liquid water in the field of atmospheric science. To calculate  $T_f$  from  $\text{RH}_l$ , we can note that  $p_v = \text{RH}_l p_v^{*,l} = \text{RH}_s p_v^{*,s}$ , and so

$$\frac{\text{RH}_s}{\text{RH}_l} = \frac{p_v^{*,l}}{p_v^{*,s}} = \left( \frac{T}{T_{\text{trip}}} \right)^{(c_{vs} - c_{vl})/R_v} \exp \left[ \frac{E_{0s} + (c_{vs} - c_{vl})T_{\text{trip}}}{R_v} \right] \times \left( \frac{1}{T} - \frac{1}{T_{\text{trip}}} \right). \quad (9)$$

This relation allows us to convert from  $\text{RH}_l$  to  $\text{RH}_s$  or vice versa, enabling the calculation of both  $T_d$  and  $T_f$  no matter which relative humidity is given.

Before we calculate the errors in expressions for the dew and frost points, let us first estimate the uncertainties from modern laboratory measurements. We can estimate the empirical uncertainty in the dewpoint by taking the reported relative uncertainty in laboratory measurements of the vapor pressure  $\Delta \log p_v^{*,l}$  and dividing by  $L_e/(R_v T^2)$ , where  $L_e$  is the latent heat of evaporation. (This estimation is used here as an alternative to propagating errors in the standard way because it is not known to what degree the reported uncertainties in the vapor pressures at  $T$  and  $T_d$  are correlated.) Similarly, we can estimate the uncertainty in the frost point using the  $\Delta \log p_v^{*,s}$  reported for laboratory measurements and dividing by  $(L_e + L_m)/(R_v T^2)$ , where  $L_m$  is the latent heat of melting. This procedure gives an uncertainty of  $\sim 0.003 \text{ K}$  at temperatures around 300 K [using  $\Delta \log p_v^{*,l} = 2 \times 10^{-4}$  from Fig. 6.4 of [Wagner and Prueß \(2002\)](#)] and that grows to  $\sim 0.02 \text{ K}$  at temperatures around 252 K [using  $\Delta \log p_v^{*,l} = 2 \times 10^{-3}$  from Table 2 of [Beltramo et al. \(2020\)](#)] and  $\sim 0.03 \text{ K}$  at temperatures around 180 K [using  $\Delta \log p_v^{*,s} = 5 \times 10^{-3}$  from Fig. 5 of [Wagner et al. \(2011\)](#)]. Note that 252 K was the lowest temperature measured by [Beltramo et al. \(2020\)](#) for supercooled water, so this is a conservative lower bound on the dewpoint uncertainty at lower temperatures. In summary, we see that uncertainties in laboratory measurements lead to uncertainties in the dew and frost points as high as a few hundredths of a degree.

As mentioned at the beginning, there is one expression for the dewpoint already in the literature that is analytic, explicit in terms of temperature and relative humidity, and derivable from theory. That equation is derived by adding to the RK approximations the further assumption that all phases of water have the same specific heat capacity, i.e.,  $c_{pv} = c_{vl} = c_{vs}$ . In that case, Eq. (3) reduces to (e.g., [Callendar 1911](#); [Murphy and Koop 2005](#))

$$p_v^{*,l} = p_{\text{trip}} \exp \left[ \frac{L_{e0}}{R_v} \left( \frac{1}{T_{\text{trip}}} - \frac{1}{T} \right) \right] \quad (c_{pv} = c_{vl} \text{ approximation}),$$

where the latent heat of evaporation is approximated as  $L_{e0} = E_{0v} + R_v T_{\text{trip}}$ . Using  $\text{RH}_l = p_v^{*,l}(T_d)/p_v^{*,l}(T)$  and solving for  $T_d$ , we get ([Lawrence 2005](#))

$$T_d = \left( \frac{1}{T} - \frac{R_v}{L_{e0}} \log \text{RH}_l \right)^{-1} \quad (c_{pv} = c_{vl} \text{ approximation}).$$

A similar expression may be obtained for  $T_f$ , replacing  $\text{RH}_l$  with  $\text{RH}_s$  and replacing the latent heat of evaporation with the approximated sum of the latent heats of evaporation and sublimation,  $L_{e0} + L_{m0} = E_{0v} + E_{0s} + R_v T_{\text{trip}}$ . When calculating

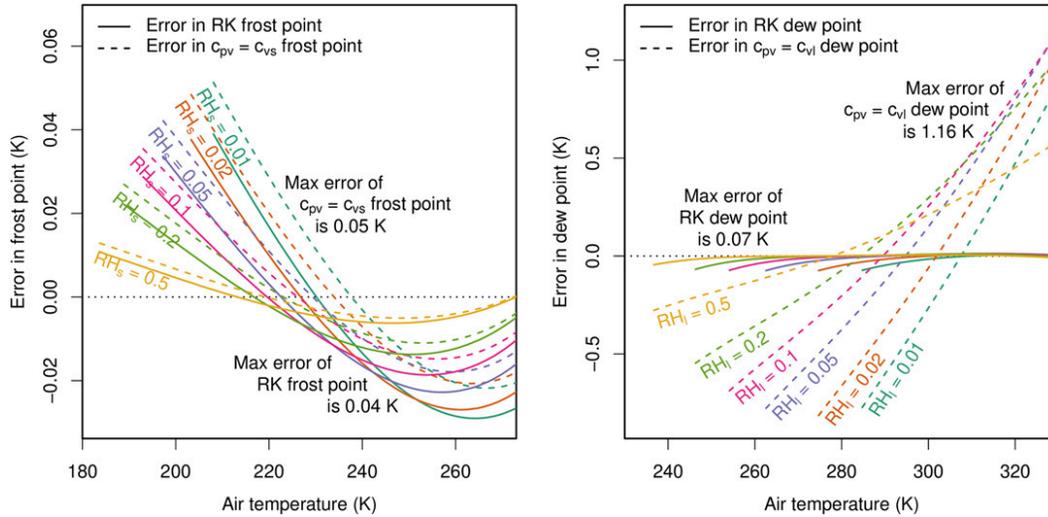


FIG. 1. Errors in the (left) frost point and (right) dewpoint as obtained from the Rankine–Kirchhoff approximations (solid) or the approximation of equal heat capacities (dashed). Note that the ordinates differ between the left and right panels. Errors are calculated relative to laboratory measurements of vapor pressure summarized by the expressions of [Murphy and Koop \(2005\)](#) for liquid and [Wagner et al. \(2011\)](#) for ice. See the text for a comparison of these errors to the uncertainties from laboratory measurements.

these expressions, the values used for  $E_{0v}$ ,  $E_{0s}$ , and  $R_v$  are the same as those given above.

The errors in these equal-heat-capacity dewpoint and frost-point expressions can be calculated by subtracting from them the dewpoints and frost points obtained (by use of a root solver) from laboratory measurements of the saturation vapor pressure. For this purpose, we will use Eq. (10) of [Murphy and Koop \(2005\)](#) for  $p_v^{*l}$ , which fits the empirical data as well or better (see Fig. 2 of [Beltramino et al. 2020](#)) than the International Association for the Properties of Water and Steam (IAPWS) G12-15 formulation ([IAPWS 2015](#)), and we will use Eq. (4) of [Wagner et al. \(2011\)](#) for  $p_v^{*s}$ , which was adopted as IAPWS R14-08 ([IAPWS 2011](#)). The dashed curves in Fig. 1 show the errors in the equal-heat-capacity frost point (left panel) and dewpoint (right panel) plotted as functions of air temperature for a variety of relative humidities. Note that the two panels use different ordinates. For the frost point, the curves are plotted for values of  $T$  such that both  $T$  and  $T_d$  lie between 180 and 273 K. For the dewpoint, the curves are plotted for values of  $T$  such that both  $T$  and  $T_d$  lie between 230 and 330 K.

The error in the equal-heat-capacity expression for the frost point is less than or equal to 0.05 K, which is the same order of magnitude as the uncertainty from laboratory measurements. For the dewpoint, however, the assumption of  $c_{pv} = c_{vl}$  generates errors that far exceed the uncertainties from empirical data. For low relative humidities (less than or equal to about 0.2) or for high air temperatures (exceeding about 310 K, 37°C, or 100°F), the dewpoint error can approach and exceed 0.5°–1.0°C or 1°–2°F.

Repeating this calculation for the Rankine–Kirchhoff expressions in Eqs. (5) and (7) gives the solid curves in Fig. 1. We see that the errors in the RK dewpoint and RK frost point are less than 0.04 and 0.07 K, respectively. These errors are the same order of magnitude as the uncertainties from the laboratory measurements. With maximum errors measured in

hundredths of a kelvin, the Rankine–Kirchhoff expressions for the dewpoint and frost point are sufficiently accurate for most, if not all, atmospheric applications.

To give a graphical summary of the Rankine–Kirchhoff dewpoint and frost-point depressions, Fig. 2 plots  $\min[T - T_d(T, RH), T - T_f(T, RH)]$  (i.e., the smaller of either the dewpoint

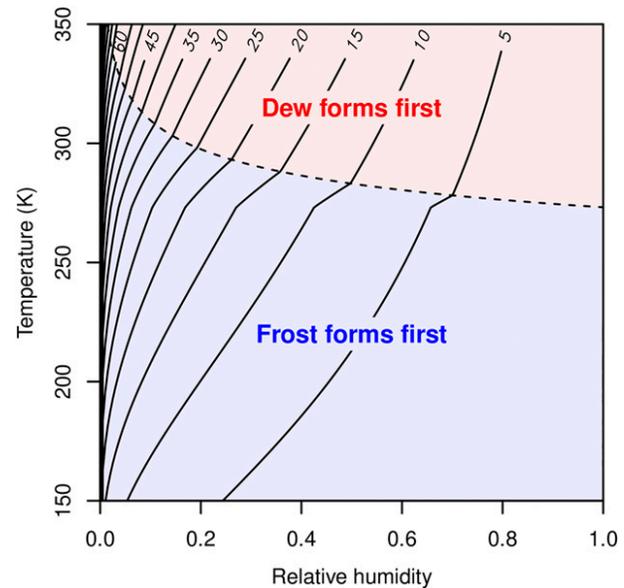


FIG. 2. Minimum of the dewpoint depression and frost-point depression plotted as a function of temperature and relative humidity (with respect to the saturation vapor pressure over liquid for  $T \geq 273.16$  K and over solid for  $T < 273.16$  K). The dashed line separates the regions where, as the air is cooled, dew forms first (red) and frost forms first (blue).

depression or the frost-point depression) as a function of temperature and relative humidity (with respect to the saturation vapor pressure over liquid for  $T \geq 273.16$  K and over solid for  $T < 273.16$  K). The kink in the curves at a temperature of 273.16 K marks the transition from relative humidity with respect to solid (below the kink) and with respect to liquid (above the kink). A second locus of kinks, marked by the dashed line, occurs where the dewpoint depression and frost-point depression are equal. Above the dashed line, dew forms before frost as the air is cooled, and vice versa below the dashed line, assuming readily available surfaces for both condensation and deposition. Code to evaluate Eqs. (5) and (7) are available on the author's website in R, Python, FORTRAN, and MATLAB.

*Acknowledgments.* This work was supported by the U.S. Department of Energy's (DOE) Atmospheric System Research (ASR), an Office of Science, Office of Biological and Environmental Research program; Lawrence Berkeley National Laboratory is operated for the DOE by the University of California under Contract DE-AC02-05CH11231. Code to calculate the Rankine–Kirchhoff dewpoint and frost point may be downloaded from <https://romps.berkeley.edu/papers/pubs-2020-dewpoint.html>.

#### REFERENCES

- August, E. F., 1828: Ueber die Berechnung der Expansivkraft des Wasserdunstes. *Ann. Phys.*, **89**, 122–137, <https://doi.org/10.1002/andp.18280890511>.
- Beltramo, G., L. Rosso, R. Cuccaro, S. Tabandeh, D. Smorgon, and V. Fericola, 2020: Accurate vapor pressure measurements of supercooled water in the temperature range between 252 K and 273 K. *J. Chem. Thermodyn.*, **141**, 105944, <https://doi.org/10.1016/j.jct.2019.105944>.
- Bosen, J. F., 1958: An approximation formula to compute relative humidity from dry bulb and dew point temperatures. *Mon. Wea. Rev.*, **86**, 486, [https://doi.org/10.1175/1520-0493\(1958\)086<0486:AAFTCR>2.0.CO;2](https://doi.org/10.1175/1520-0493(1958)086<0486:AAFTCR>2.0.CO;2).
- Callendar, H. L., 1911: Vaporization. *The Encyclopaedia Britannica: A Dictionary of Arts, Sciences, Literature and General Information*, 11th ed. Vol. 27, Cambridge University Press, 897–904.
- Corless, R. M., G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, 1996: On the Lambert  $W$  function. *Adv. Comput. Math.*, **5**, 329–359, <https://doi.org/10.1007/BF02124750>.
- Dalton, J., 1802a: Meteorological observations. *Mem. Lit. Philos. Soc. Manchester*, **5**, 666–674.
- , 1802b: On the constitution of mixed gases; on the force of steam or vapour from water and other liquids in different temperatures, both in a Torricellian vacuum and in air; on evaporation; and on the expansion of gases by heat. *Mem. Lit. Philos. Soc. Manchester*, **5**, 535–602.
- , 1824: Observations in meteorology, particularly with regard to the dew-point, or quantity of vapour in the atmosphere; made on the mountains in the north of England. *Mem. Lit. Philos. Soc. Manchester*, **4**, 104–124.
- Emanuel, K. A., 1994: *Atmospheric Convection*. Oxford University Press, 580 pp.
- Hauf, T., and H. Höller, 1987: Entropy and potential temperature. *J. Atmos. Sci.*, **44**, 2887–2901, [https://doi.org/10.1175/1520-0469\(1987\)044<2887:EAPT>2.0.CO;2](https://doi.org/10.1175/1520-0469(1987)044<2887:EAPT>2.0.CO;2).
- Howard, L., 1818: *The Climate of London, Deduced from Meteorological Observations, Made at Different Places in the Neighbourhood of the Metropolis*. W. Phillips, 358 pp.
- IAPWS, 2011: Revised release on the pressure along the melting and sublimation curves of ordinary water substance. IAPWS R14-08(2011), 7 pp., <http://www.iapws.org/relguide/MeltSub.html>.
- , 2015: Guideline on thermodynamic properties of supercooled water. IAPWS G12-15, 10 pp., <http://www.iapws.org/relguide/Supercooled.html>.
- Kirchhoff, G., 1858: Ueber die Spannung des Dampfes von Mischungen aus Wasser und Schwefelsäure. *Ann. Phys. Chem.*, **180**, 612–621, <https://doi.org/10.1002/andp.18581800806>.
- Lawrence, M. G., 2005: The relationship between relative humidity and the dewpoint temperature in moist air: A simple conversion and applications. *Bull. Amer. Meteor. Soc.*, **86**, 225–234, <https://doi.org/10.1175/BAMS-86-2-225>.
- Magnus, G., 1844: Versuche über die Spannkraft des Wasserdampfes. *Ann. Phys. Chem.*, **137**, 225–247, <https://doi.org/10.1002/andp.18441370202>.
- Mitchell, L. V., 1967: Radiosonde dew-point accuracies 40°C to –40°C. U.S. Air Force Tech. Rep. 198, 27 pp., <https://apps.dtic.mil/dtic/tr/fulltext/u2/659760.pdf>.
- Murphy, D. M., and T. Koop, 2005: Review of the vapour pressures of ice and supercooled water for atmospheric applications. *Quart. J. Roy. Meteor. Soc.*, **131**, 1539–1565, <https://doi.org/10.1256/qj.04.94>.
- Rankine, W. J. M., 1866: On saturated vapours. *Proc. Roy. Soc. Edinburgh*, **31**, 199–201.
- Romps, D. M., 2015: MSE minus CAPE is the true conserved variable for an adiabatically lifted parcel. *J. Atmos. Sci.*, **72**, 3639–3646, <https://doi.org/10.1175/JAS-D-15-0054.1>.
- , 2017: Exact expression for the lifting condensation level. *J. Atmos. Sci.*, **74**, 3891–3900, <https://doi.org/10.1175/JAS-D-17-0102.1>.
- , and Z. Kuang, 2010: Do undiluted convective plumes exist in the upper tropical troposphere? *J. Atmos. Sci.*, **67**, 468–484, <https://doi.org/10.1175/2009JAS3184.1>.
- Sargent, G. P., 1980: Computation of vapour pressure, dew-point and relative humidity from dry- and wet-bulb temperatures. *Meteor. Mag.*, **109**, 238–246.
- Urbank, T. M., S. M. Kelly, T. O. King, and C. A. Archibald, 2001: Development and application of an integrated dew point and glass temperature sensor. SAE Tech. Paper Series 2001-01-0585, 13 pp.
- Wagner, W., and A. Pruß, 2002: The IAPWS formulation 1995 for the thermodynamic properties of ordinary water substance for general and scientific use. *J. Phys. Chem. Ref. Data*, **31**, 387–535, <https://doi.org/10.1063/1.1461829>.
- , T. Riethmann, R. Feistel, and A. H. Harvey, 2011: New equations for the sublimation pressure and melting pressure of H<sub>2</sub>O ice Ih. *J. Phys. Chem. Ref. Data*, **40**, 043103, <https://doi.org/10.1063/1.3657937>.
- Wanielista, M., R. Kersten, and R. Eaglin, 1997: *Hydrology: Water Quantity and Quality Control*. 2nd ed. John Wiley and Sons, 567 pp.