

MSE Minus CAPE is the True Conserved Variable for an Adiabatically Lifted Parcel

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ABSTRACT

For an adiabatic parcel convecting up or down through the atmosphere, it is often assumed that its moist static energy (MSE) is conserved. Here, it is shown that the true conserved variable for this process is MSE minus convective available potential energy (CAPE) calculated as the integral of buoyancy from the parcel's height to its level of neutral buoyancy and that this variable is conserved even when accounting for full moist thermodynamics and nonhydrostatic pressure forces. In the calculation of a dry convecting parcel, conservation of MSE minus CAPE gives the same answer as conservation of entropy and potential temperature, while the use of MSE alone can generate large errors. For a moist parcel, entropy and equivalent potential temperature give the same answer as MSE minus CAPE only if the parcel ascends in thermodynamic equilibrium. If the parcel ascends with a nonisothermal mixed-phase stage, these methods can give significantly different answers for the parcel buoyancy because MSE minus CAPE is conserved, while entropy and equivalent potential temperature are not.

1. Introduction

One of the most fundamental and ubiquitous calculations in atmospheric science is the calculation of the properties of an adiabatically lifted air parcel—that is, a parcel lifted adiabatically, vertically, and rapidly enough so that the environment through which it rises can be considered time invariant. This calculation is performed thousands of times per day at weather centers around the world to quantify atmospheric instability and storm potential. It is also calculated many millions of times per day on supercomputers that are forecasting next week's weather and next century's climate. Despite the importance of this process, there is no agreement on how it should be calculated. The most common approach is to use conservation of moist static energy (MSE), which is defined as the sum of sensible enthalpy, latent enthalpy, and gravitational potential energy; see Eqs. (5) and (6) below for precise expressions. But, it is widely known that MSE is only approximately conserved for an adiabatically lifted parcel, and there seems to be no consensus on what alternative should be used.

An alternative that we explore here is the difference between MSE and convective available potential energy (CAPE) calculated as the integral of parcel buoyancy from the parcel's height to its level of neutral buoyancy (LNB). Riehl and Malkus (1958) were the first to derive the conservation of MSE minus CAPE (MSE – CAPE) for an adiabatically lifted parcel [see their Eq. (10)] but the approximations in that derivation neglected the effect of water phase on density, pressure, and heat capacity. They also stated that MSE – CAPE is approximately conserved only when the parcel buoyancy is small; as shown here, this is incorrect.

Several years later, Madden and Robitaille (1970) and Betts (1974) observed that the integral of a parcel's buoyancy can explain the difference in the level of its neutral buoyancy predicted using conservation of MSE versus conservation of equivalent potential temperature θ_e . Levine (1972) tried to refute this claim, but did so using a fallacious derivation (Madden and Robitaille 1972). Most importantly, none of these derivations accounted for the virtual effects of water and the dependence of heat capacity on composition.

The goal of this note is to show, with the full effects of water included, that MSE – CAPE is conserved for an adiabatically lifted parcel so long as its buoyancy is defined with respect to the hydrostatic state of the mean large-scale environment. We will show that this conservation is exact

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even in the presence of nonhydrostatic pressure-perturbation forces that convert CAPE to turbulent energy and wave energy in the environment. While entropy and equivalent potential temperature are conserved for adiabatic parcels in thermodynamic equilibrium, we emphasize that MSE – CAPE is conserved for adiabatic parcels that are either in equilibrium (e.g., a single condensed phase) or in disequilibrium (e.g., a nonisothermal mixed-phase stage, as occurs during the Bergeron process). In contrast, entropy and equivalent potential temperature are not conserved for disequilibrium processes. We argue, therefore, that MSE – CAPE is the true conserved variable for an adiabatically lifted parcel and that it—and not MSE, entropy, or equivalent potential temperature—should be used to calculate the parcel's properties.

2. Theory

For a moist atmosphere, the governing equation for internal energy (i.e., the first law of thermodynamics) is

$$\rho \frac{d}{dt} [c_{vm}(T - T_{\text{trip}}) + q_v E_{0v} - q_s E_{0s}] = -p \nabla \cdot \mathbf{u} + Q, \quad (1)$$

where p is the pressure, \mathbf{u} is the velocity, and Q is the external heating with dimensions of power per volume. Here, c_{vm} is the heat capacity at constant volume for moist air, which is given in terms of mass fractions and heat capacities of dry air (subscript a), water vapor (v), liquid water (l), and solid water (s) as $c_{vm} = q_a c_{va} + q_v c_{vv} + q_l c_{vl} + q_s c_{vs}$. Since the term in square brackets is the specific internal energy, we see that the constant E_{0v} is the difference in specific internal energy between water vapor and liquid at the triple-point temperature ($T_{\text{trip}} = 273.16 \text{ K}$) and E_{0s} is the difference in specific internal energy between water liquid and solid at the triple-point temperature.

Adding $\rho d(p/\rho)/dt$ to both sides turns the $\rho d(c_{vm}T)/dt$ into $\rho d(c_{pm}T)/dt$, where c_{pm} is the heat capacity at constant pressure for moist air, and, with help from the continuity equation ($\partial \rho / \partial t + \nabla \cdot \rho \mathbf{u} = 0$), turns the $-p \nabla \cdot \mathbf{u}$ into dp/dt . Since the specific gas constant of moist air R_m equals $q_a R_a + q_v R_v$, and since dq_a/dt is zero for an adiabatic parcel, the Lagrangian derivative of $R_m T_{\text{trip}} - q_v R_v T_{\text{trip}}$ is zero. Subtracting this form of zero from the left-hand side then gives

$$\rho \frac{d}{dt} [c_{pm}(T - T_{\text{trip}}) + q_v(E_{0v} + R_v T_{\text{trip}}) - q_s E_{0s}] = \frac{dp}{dt} + Q. \quad (2)$$

This is the governing equation for enthalpy. Note that $E_{0v} + R_v T_{\text{trip}}$ is the difference in specific enthalpy between water vapor and liquid at the triple-point temperature.

Let us now consider a parcel that is ascending through an environment whose large-scale mean pressure and density are steady and hydrostatic. Let us denote that large-scale environmental pressure and density by $p_e(z)$ and $\rho_e(z)$; by hydrostatic balance, $\partial p_e / \partial z = -\rho_e g$. Note that we do not require the flow to be hydrostatic at small scales or in the vicinity of the parcel. Instead, we allow there to be substantial pressure perturbations p' , so that $p = p_e + p'$. This means that there may be significant pressure-perturbation forces ($-\nabla p'$) on the parcel. We will, however, assume that the pressure at the location of the parcel is equal to p_e ; this is the usual assumption made when calculating the properties of a lifted parcel. With these two assumptions, we can write dp/dt as $dp_e/dt = w \partial p_e / \partial z = -w g \rho_e$. Assuming the parcel is adiabatic ($Q = 0$), dividing by ρw , and writing $g \rho_e / \rho$ as $b + g$, where $b = g(\rho_e / \rho - 1)$ is the parcel's buoyancy, we get

$$\frac{d}{dz} (\text{MSE} - \text{CAPE}) = 0, \quad (3)$$

where CAPE is the integral of buoyancy from the height z of the parcel to some fixed reference height z_{top} ,

$$\text{CAPE} = \int_z^{z_{\text{top}}} dz' b, \quad (4)$$

and MSE can be defined in one of two equivalent ways:

$$\text{MSE} = c_{pm}(T - T_{\text{trip}}) + q_v(E_{0v} + R_v T_{\text{trip}}) - q_s E_{0s} + gz, \quad \text{or} \quad (5)$$

$$\text{MSE} = [q_a c_{pa} + (q_v + q_l + q_s) c_{vl}] (T - T_{\text{trip}}) + q_v L_c - q_s L_f + gz, \quad (6)$$

where $L_c = E_{0v} + R_v T + (c_{vv} - c_{vl})(T - T_{\text{trip}})$ is the latent heat of condensation and $L_f = E_{0s} + (c_{vl} - c_{vs})(T - T_{\text{trip}})$ is the latent heat of fusion. The implication of Eq. (3) is that an adiabatic parcel's MSE and CAPE decrease with height at the same rate. Related expressions have been obtained in the anelastic approximation applied to the ocean (Ingersoll 2005) and the atmosphere (Pauluis 2008).

The upper bound of integration in Eq. (4) is somewhat arbitrary. To be as consistent as possible with standard definitions of CAPE, it can be chosen to be a level of neutral buoyancy. For our purposes, all that really matters is that the upper bound is a constant—that is, that it does not change as the parcel is lifted. Note that Eq. (3) is used to calculate parcel ascent, and a constant upper bound drops out in that equation. Also, note that there is no unit step function of buoyancy in the integrand of Eq. (4) that would restrict the integral to

regions of positive buoyancy, as is often the case in other definitions of CAPE.

The assumptions used in deriving Eq. (3) are no more restrictive than the assumptions used when calculating parcel properties with conservation of equivalent potential temperature θ_e . When using θ_e to calculate the parcel's temperature at some height, the pressure of the parcel at that height must be known, and it is always assumed that $p = p_e$. As for the assumption of large-scale hydrostatic balance, this is essentially guaranteed for averages of environmental pressure and density taken over scales much larger than the scale height. Note that $MSE - CAPE$ is conserved even if the parcel does not convert CAPE to its own kinetic energy (KE), but, instead, dissipates CAPE to environmental turbulence and wave energy. In other words, $MSE + KE$ is not conserved, but $MSE - CAPE$ is.

Why is MSE itself not conserved? Consider Eq. (2) for an adiabatic parcel. Dividing by ρ , the term $(1/\rho)dp/dt$ can be written as the term gz in MSE only if the density of the parcel ρ equals the density of the mean large-scale environment ρ_e . If the parcel is buoyant (i.e., $1/\rho > 1/\rho_e$), then $(1/\rho)dp/dt$ is larger in magnitude than $(1/\rho_e)dp/dt$, and this saps more specific enthalpy from the parcel than would be predicted by geopotential alone. In other words, for a given change in pressure, a parcel loses more specific enthalpy the lighter it is.

3. Examples

The adiabatic ascent of an air parcel is typically calculated using either conservation of MSE, entropy, or θ_e . Although θ_e has been written in many different ways with varying degrees of completeness and accuracy (e.g., Simpson 1978; Bolton 1980; Hauf and Höller 1987; Marquet 2011), θ_e is simply the exponential of entropy (Romps and Kuang 2010). Therefore, there are really just two distinct methods: conservation of MSE and conservation of entropy. We will now consider four examples that illustrate the errors in parcel buoyancy that are caused by incorrectly assuming that MSE is conserved or, for a parcel with disequilibrium mixed-phase microphysics, by assuming that entropy is conserved.

a. Dry case with constant-lapse-rate environment

Consider a dry environment with a dry-adiabatic lapse rate. We can calculate the temperature excess for a dry parcel rising through that environment using conservation of MSE minus CAPE:

$$\frac{d}{dz}(MSE - CAPE) = 0, \tag{7}$$

$$MSE = c_{pa}T + gz, \quad \text{and} \tag{8}$$

$$CAPE = \int_z^{z_{\text{top}}} dz' g \frac{T - T_e}{T_e}. \tag{9}$$

Denoting the environment's surface air temperature by T_0 , the environmental temperature profile is $T_e = T_0 - gz/c_{pa}$. Note that this solution satisfies Eq. (7) with $CAPE = 0$, confirming that g/c_{pa} is the lapse rate for an atmosphere in which displaced parcels are neutrally buoyant. For $g = 10 \text{ m s}^{-2}$, $c_{pa} = 1000 \text{ J kg}^{-1} \text{ K}^{-1}$, and $T_0 = 300 \text{ K}$, the temperature of the environment decreases linearly with height from 300 K at the surface to 0 K at the top of the atmosphere, which is at 30 km.

Now, consider a buoyant parcel in this environment. If we mistakenly used conservation of MSE, then we would conclude that the parcel's lapse rate is, like the environment, equal to g/c_{pa} , thereby maintaining a constant temperature excess as it rises. This is incorrect. Defining the temperature excess of a buoyant parcel as $\Delta T = T - T_e$, we can write Eq. (7) for the parcel as

$$c_{pa} \frac{d}{dz} \Delta T + b = 0.$$

This tells us that the parcel's temperature excess decreases with height. Since $b = g\Delta T/T_e$, we can solve for ΔT to find

$$\Delta T(z) = \Delta T_0 \left(1 - \frac{gz}{c_{pa}T_0} \right),$$

where ΔT_0 is the parcel's temperature excess at the surface. Therefore, we see that the parcel's temperature excess decreases linearly with height from ΔT_0 at the surface to zero at the top of the atmosphere. For example, at 15 km, ΔT is exactly half of its original value. This case is illustrated in the left panel of Fig. 1 for a large ΔT_0 . This choice of ΔT_0 is made so that both the temperature profiles and the parcel buoyancies are readily visible without the need for a skew- T axis; nothing about Fig. 1 changes qualitatively for a smaller ΔT_0 .

For a dry parcel, conservation of $MSE - CAPE$ must give the same answer as conservation of potential temperature. This is easy to prove for this example. Both the environment and the parcel have a constant potential temperature (i.e., independent of height). At any given height, the parcel and environment have the same pressure, so the ratio of the parcel's potential temperature and the environment's potential temperature is simply $(T + \Delta T)/T = 1 + \Delta T/T$. Since the potential temperatures are constant, the ratio $\Delta T/T$ must be constant. Since T goes to zero linearly with height, so

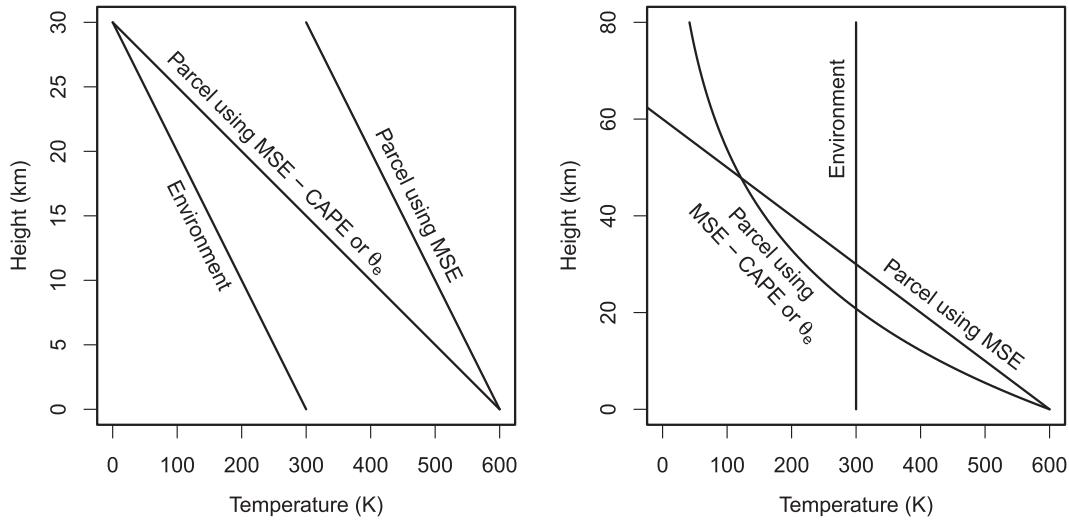


FIG. 1. (left) Temperature profiles for an environment with a surface temperature of 300 K and a lapse rate of g/c_{pa} , an adiabatic parcel with a temperature at the surface of 600 K calculated correctly using conservation of MSE – CAPE or conservation of θ_e , and the temperature of the same parcel calculated incorrectly using conservation of MSE. (right) As in (left), but for an isothermal atmosphere with a temperature of 300 K.

must ΔT . This completes the proof. In Fig. 1, the potential temperature is written as the equivalent potential temperature θ_e since θ_e equals θ for a dry parcel; the appendix provides the precise definition of θ_e .

b. Dry case with isothermal environment

Let us now consider a dry parcel rising through a dry, isothermal atmosphere. Since the air is dry, Eqs. (7)–(9) still hold. Since the environment is isothermal, T_e is independent of height and we can write Eq. (7) for the parcel as

$$c_{pa} \frac{d}{dz} \Delta T + g + \frac{g}{T_e} \Delta T = 0.$$

The solution to this equation is

$$\Delta T = (T_e + \Delta T_0) e^{-gz/c_{pa} T_e} - T_e, \quad (10)$$

where ΔT_0 is the parcel's temperature excess at the surface. We see that the temperature excess decreases with height, but not linearly. This solution is illustrated in the right panel of Fig. 1. For a parcel that is buoyant at the surface (i.e., $\Delta T_0 > 0$), then the level of neutral buoyancy is

$$z_{\text{LNB}} = \frac{c_{pa} T_e}{g} \log \left(1 + \frac{\Delta T_0}{T_e} \right). \quad (11)$$

For $\Delta T_0 \ll T_e$, $z_{\text{LNB}} \approx c_{pa} \Delta T_0 / g$. In general, the parcel's temperature decays toward zero with a scale height of $c_{pa} T_e / g$. We can compare this to the assumption of conservation of MSE alone, which would predict

$\Delta T = \Delta T_0 - gz/c_{pa}$ as well as the nonsensical result that the parcel's absolute temperature would be negative above a height equal to $c_{pa} g$ times its temperature at the surface.

Of course, we can confirm in this case that conservation of MSE – CAPE gives the same result as conservation of potential temperature. Since the environmental pressure profile is $p_e = p_0 \exp(-gz/R_a T_e)$, where p_0 is the surface pressure, the parcel's potential temperature at height z is $\theta = (T_e + \Delta T)(p_0/p_e)^{R_a/c_{pa}} = (T_e + \Delta T) e^{gz/c_{pa} T_e}$. Since $\theta = T_e + \Delta T_0$, this can be solved to give Eq. (10).

c. Moist case without ice

For the first of two examples with moisture, we will simplify things by eliminating the solid phase of water. This is effectively accomplished by setting c_{vs} equal to c_{vl} and setting E_{0s} to zero in the definition of MSE and in the equations in the appendix. For the environmental profile, we choose a moist adiabat with $p = 1$ bar, $T = 300$ K, and $\text{RH} = 1$ at $z = 0$. This environmental profile is calculated using conservation of MSE – CAPE with CAPE set to zero (so that a parcel of air displaced from the environment will have zero buoyancy). In this case, since there is no ice and a thermodynamic equilibrium between water vapor and liquid is strictly obeyed, conservation of moist entropy gives the same environmental profile.

Next, consider a buoyant, saturated parcel of air that starts at the surface with $T = 300.5$ K. The left panel of Fig. 2 shows the profile of buoyancy (expressed as a virtual-temperature anomaly) for this parcel as it is lifted, calculated in three different ways: using conservation of MSE – CAPE (solid), conservation of moist entropy

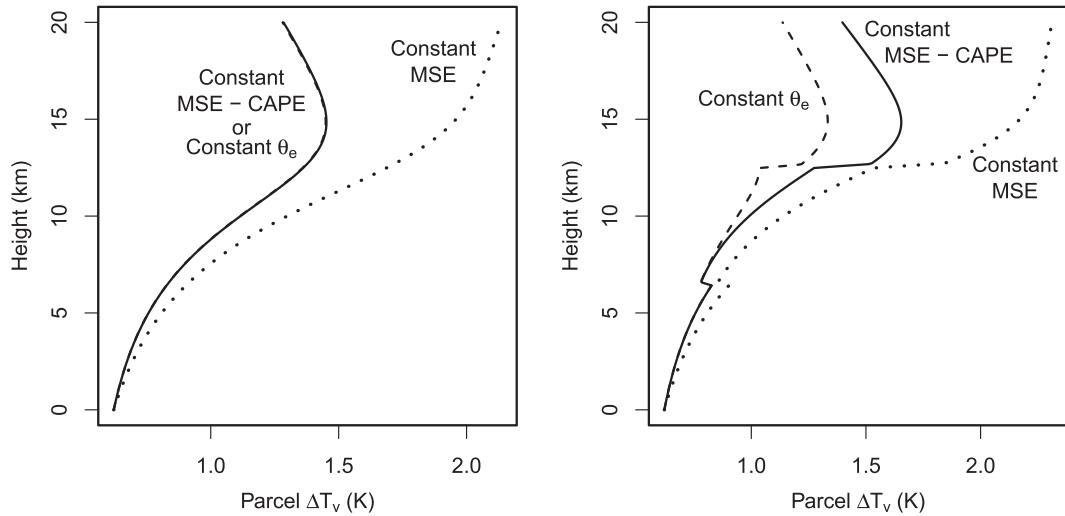


FIG. 2. (left) Profiles of buoyancy for a buoyant moist parcel rising adiabatically through a moist-adiabatic environment in which neither has solid water (so that there is no mixed-phase thermodynamic disequilibrium). The profiles are shown as calculated correctly using either conservation of MSE – CAPE (solid) or conservation of θ_e (dashed) or calculated incorrectly using conservation of MSE (dotted). (right) As in the left panel, but for an atmosphere in which parcels have a mixed-phase stage between a temperature of 273.16 K and a temperature of 240 K. In this case, entropy is not conserved, so calculating the parcel ascent using conservation of entropy gives the wrong answer.

(dashed; overlapping the solid curve, so it is difficult to see), and conservation of MSE (dotted). Since the parcel is always in thermodynamic equilibrium, conservation of θ_e gives the correct answer, as seen from the overlap of the dashed and solid curves. Conservation of MSE, however, overestimates the buoyancy of the parcel by as much as 1 K, which is substantial for a parcel whose buoyancy is in the range of 0.5–1.5 K.

d. Moist case with mixed-phase condensates

We now include ice by using the correct values for c_{vs} and E_{0s} . If a cloud parcel starts at a warm cloud base and rises through the atmosphere in thermodynamic equilibrium, then it experiences three stages. It begins in a state of thermodynamic equilibrium between water vapor and liquid. Then, at a temperature of 273.16 K, the parcel begins a stage of isothermal ascent in which vapor, liquid, and ice are in thermodynamic equilibrium at the triple point. The depth of this isothermal stage is approximately $q_l E_{0s}/g$, where q_l is the mass fraction of liquid water when the parcel first hits a temperature of 273.16 K. For $q_l = 0.02$, this is a depth of 600 m. At the end of the isothermal stage, all of the condensates have been converted to ice and the parcel enters its third stage, in which vapor and ice are in equilibrium.

In reality, this is not how the physics of clouds works. Instead of a ~600-m isothermal mixed-phase stage, there is a multikilometer nonisothermal mixed-phase stage in which vapor, liquid, and ice are in disequilibrium, typically with the vapor mass fraction in between saturation with

respect to liquid and saturation with respect to ice. This causes vapor to diffuse down a vapor gradient, and that diffusion is a source of entropy (Pauluis and Held 2002). Therefore, adiabatic parcels that ascend with a multi-kilometer mixed-phase region do not conserve entropy. This means that entropy (or, equivalent potential temperature) should not be used to calculate adiabatic parcel ascent. Fortunately, MSE – CAPE is conserved in these cases. MSE – CAPE is conserved because it is based on conservation of energy, and the only sources and sinks of energy for an adiabatic parcel are pressure work (both expansion work and work against the pressure-perturbation drag force), and both of those are properly accounted for in MSE – CAPE no matter how out of thermodynamic equilibrium the parcel is. Therefore, MSE – CAPE is the true conserved variable for an ascending adiabatic parcel.

The right panel of Fig. 2 illustrates the failure of MSE conservation and θ_e conservation in the case of a parcel with a disequilibrium mixed-phase stage. The environmental profile is calculated from conservation of MSE – CAPE with CAPE set to zero (so that displaced environmental parcels are neutrally buoyant) and $p = 1$ bar, $T = 300$ K, and $RH = 1$ at $z = 0$. The mixed-phase region is defined to start at 273.16 K and end at 240 K. In between those two temperatures, the fraction of condensates in the ice phase transitions to one as a linear function of temperature. Similarly, the vapor mass fraction is set to a linear weighting of liquid and solid saturation vapor mass fractions; see the appendix for the

relevant equations. The lifted parcel is subjected to these same rules of mixed-phase disequilibrium, and the parcel is initialized at $z = 0$ with $T = 300.5$ K and $\text{RH} = 1$.

As seen in the right panel of Fig. 2, the profiles of buoyancy calculated using conservation of MSE and conservation of θ_e are both erroneous, deviating from the correct profile (calculated by conservation of MSE – CAPE) by as much as 0.3–0.9 K. The profile calculated using conservation of MSE gives buoyancy that is too high because it misses the reduction of MSE due to CAPE. The profile of buoyancy calculated using conservation of entropy gives buoyancy that is too low because it misses the fact that the mixed-phase disequilibrium increases entropy. Note that the profile of buoyancy calculated by conservation of θ_e is correct up until the beginning of the mixed-phase layer, which begins around 6 km.

4. Summary

We have shown that MSE – CAPE, with CAPE calculated as the integral of parcel buoyancy up to the parcel's height, is conserved for an adiabatically lifted parcel. The assumptions going into this conservation law are no more restrictive than the assumptions usually made for parcel calculations. In particular, the buoyancy must be calculated with respect to the density profile of the mean large-scale environment, which, by virtue of being large scale, is hydrostatic. As in any other method for calculating parcel properties, we assume that the pressure of the parcel is equal to the mean large-scale environmental pressure at the same height. Aside from this equality of pressures, no assumption is made about the pressure perturbations; arbitrary pressure-perturbation forces (e.g., form drag and wave drag) are allowed and have no impact on the conservation of MSE – CAPE. In addition, no assumption is made about thermodynamic equilibrium. Thermodynamic disequilibrium, such as a nonisothermal mixed-phase stage, will alter a parcel's entropy, but has no effect on MSE – CAPE.

Note that calculating parcel ascent using the conservation of MSE – CAPE is no more difficult than calculating parcel ascent using (the incorrect) conservation of MSE. Taking d/dz of MSE – CAPE and setting it to zero, we get

$$\frac{d}{dz} \text{MSE} = -b, \quad (12)$$

where MSE is the moist static energy of the parcel at height z , given by either Eq. (5) or (6), and b is the buoyancy of the parcel at height z . If we know the state of the parcel at height z (i.e., its temperature and its mixing ratios of water vapor, liquid, and solid), then we know MSE and b . Equation (12) then tells us the MSE at

height $z + \Delta z$, for some small Δz . Using pressure equality of the parcel and its environment, along with whatever assumptions we are making about the apportionment of water among the three phases [e.g., a mixed-phase transition from liquid condensates to solid condensates as a linear function of temperature as in Romps and Kuang (2010)], we can use this value of MSE to calculate the parcel's temperature at $z + \Delta z$. We then know its buoyancy and can proceed to $z + 2\Delta z$. With this simple procedure, we can integrate the parcel's properties upward in height.

Although the discussion in this paper has focused on adiabatic parcels, these results are just as important for entraining parcels or parcels that lose condensates by fallout. In the numerical calculation of such parcels, the ascent or descent is typically split into a sequence of serial processes. For example, the calculation of an entraining parcel with condensate fallout may proceed as a loop over the following processes with one iteration for each small height increment: entrainment at constant pressure, fallout at constant pressure, and adiabatic lifting. Regardless of the processes performed at constant pressure, the lifting from one pressure to the next should be performed using conservation of MSE – CAPE.

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APPENDIX

Moist Thermodynamics

The following definition of equivalent potential temperature,

$$\theta_e = T \left(\frac{p_0}{p_a} \right)^{R_a/c_{pa}} \left(\frac{T}{T_{\text{trip}}} \right)^{(r_v c_{pv} + r_l c_{vl} + r_s c_{vs})/c_{pa}} \times \left(\frac{p_{\text{trip}}}{p_v} \right)^{r_v R_v/c_{pv}} e^{(r_v s_{0v} - r_s s_{0s})/c_{pa}}, \quad (\text{A1})$$

is exactly conserved when moist entropy is conserved (Romps and Kuang 2010). Here, r_v , r_l , and r_s are the mixing ratios (i.e., mass per mass of dry air) of vapor, liquid,

and solid, respectively. The constant $s_{0v} = E_{0v}/T_{\text{trip}} + R_v$ is the difference in specific entropy between water vapor and liquid at the triple point, and $s_{0s} = E_{0s}/T_{\text{trip}}$ is the difference in specific entropy between water liquid and solid at the triple point.

By equating the Gibb's free energies of water vapor and liquid, the saturation vapor pressure over liquid is found to be (Romps 2008)

$$p_v^{*l} = p_{\text{trip}} \left(\frac{T}{T_{\text{trip}}} \right)^{(c_{pv} - c_{vl})/R_v} \times \exp \left[\frac{E_{0v} - (c_{vv} - c_{vl})T_{\text{trip}}}{R_v} \left(\frac{1}{T_{\text{trip}}} - \frac{1}{T} \right) \right]. \quad (\text{A2})$$

By equating the Gibb's free energies of water vapor and solid, the saturation vapor pressure over solid is found to be

$$p_v^{*s} = p_{\text{trip}} \left(\frac{T}{T_{\text{trip}}} \right)^{(c_{pv} - c_{vs})/R_v} \times \exp \left[\frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\text{trip}}}{R_v} \left(\frac{1}{T_{\text{trip}}} - \frac{1}{T} \right) \right]. \quad (\text{A3})$$

We can then rearrange Eqs. (A2) and (A3) to give expressions for the saturation vapor mass fraction as a function of p , T , and total water ($q_v + q_l + q_s$),

$$q_v^{*l} = (1 - q_v - q_l - q_s) \frac{R_a}{R_v} \left\{ \frac{p}{p_{\text{trip}}} \left(\frac{T_{\text{trip}}}{T} \right)^{(c_{pv} - c_{vl})/R_v} \times \exp \left[-\frac{E_{0v} - (c_{vv} - c_{vl})T_{\text{trip}}}{R_v} \left(\frac{1}{T_{\text{trip}}} - \frac{1}{T} \right) \right] - 1 \right\}^{-1} \quad \text{and} \quad (\text{A4})$$

$$q_v^{*s} = (1 - q_v - q_l - q_s) \frac{R_a}{R_v} \left\{ \frac{p}{p_{\text{trip}}} \left(\frac{T_{\text{trip}}}{T} \right)^{(c_{pv} - c_{vs})/R_v} \times \exp \left[-\frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\text{trip}}}{R_v} \left(\frac{1}{T_{\text{trip}}} - \frac{1}{T} \right) \right] - 1 \right\}^{-1}. \quad (\text{A5})$$

In sections 3c and 3d, the parcel is given a non-isothermal mixed-phase stage by defining its mass fractions by

$$\begin{aligned} q_v^* &= [1 - \xi(T)]q_v^{*l} + \xi(T)q_v^{*s}, \\ q_l &= [1 - \xi(T)](q_l + q_s), \quad \text{and} \\ q_s &= \xi(T)(q_l + q_s), \end{aligned}$$

with ξ prescribing a linear transition from liquid condensates to solid condensates between $T_{\text{trip}} = 273.16$ K and 240 K:

$$\xi = \begin{cases} 1 & T \leq 240 \text{ K} \\ \frac{T_{\text{trip}} - T}{T_{\text{trip}} - 240 \text{ K}} & 240 \text{ K} < T < T_{\text{trip}} \\ 0 & T \geq T_{\text{trip}} \end{cases}.$$

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