The Dry-Entropy Budget of a Moist Atmosphere

DAVID M. ROMPS
Harvard University, Cambridge, Massachusetts

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ABSTRACT

The entropy budget has been a popular starting point for theories of the work, or dissipation, performed by moist atmospheres. For a dry atmosphere, the entropy budget provides a theory for the dissipation in terms of the imposed diabatic heat sources. For a moist atmosphere, the difficulties in quantifying irreversible moist processes or the value of the condensation temperature have so far frustrated efforts to construct a theory of dissipation. With this complication in mind, one of the goals here is to investigate the predictive power of the budget of dry entropy (i.e., the heat capacity times the logarithm of potential temperature).

Toward this end, the dry-entropy budget is derived for an atmosphere with realistic heat capacities and a solid-water phase, features that were absent from some previous studies of atmospheric entropy. It is shown that the dry-entropy budget may be interpreted as the sum of sources and sinks from six processes, which are, in order of decreasing magnitude, radiative cooling, condensation heating, sensible heating at the surface, wind-generated frictional dissipation, lifting of water, and transport of heat from the melting line to the upper troposphere. This picture leads to an alternative explanation for the low efficiency of the moist atmospheric engine.

Numerical simulations are presented from a new cloud-resolving model, Das Atmosphärische Modell, which was designed to conserve energy and close the dry-entropy budget. Simulations with and without subgrid diffusion of heat and water are compared to investigate the impact of subgrid parameterizations on the terms in the dry-entropy budget. The numerical results suggest a particularly simple parameterization of wind-generated dissipation that appears to be valid for changes in sea surface temperature and mean wind. The dry-entropy budget also points to various changes in forcings and parameterizations that could be expected to increase or decrease the wind-generated dissipation.

1. Introduction

To motivate the study of the entropy budget, consider first an enclosed, dry atmosphere. For an enclosed atmosphere in a steady state, the sum of all the entropy sources must be zero (here, "sources" is shorthand for sources and sinks). In the case of an enclosed, dry atmosphere, all of the entropy sources are simply heat sources divided by the temperature. For example, possible heat sources include radiation (Q), conduction of heat ($-\nabla \cdot \mathbf{J}$, where $\mathbf{J}$ is the conductive heat flux), and the anemonal dissipation ($\mathcal{D}$).[1] A succinct statement of the steady-state entropy budget for a dry atmosphere is

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int dx\ dy\ dz\ \left(\frac{Q}{T} - \frac{\nabla \cdot \mathbf{J}}{T} + \frac{\mathcal{D}}{T}\right) = 0, \quad (1)$$

where the spatial integral is taken over a closed volume (i.e., $\mathbf{u} = 0$ at the spatial boundaries). From here on, this time-averaged volume integral will be denoted by a subscript V,

$$\int_V \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int dx\ dy\ dz, \quad (1)$$

and an integral with a subscript S will denote a time-averaged integral over the surface boundary. Integrating by parts, the dry-atmospheric entropy budget becomes

$$\int_V \left(\frac{Q}{T} - \frac{\mathbf{J} \cdot \nabla T}{T^2} + \frac{\mathcal{D}}{T}\right) + \int_S \frac{J^z}{T} = 0, \quad (2)$$

[1] Here and throughout, the term "anemonal dissipation" refers to the heating caused by the viscous dissipation of eddies cascading down from the macroscopic motions of the atmosphere. In contrast, the term "precipitation dissipation" refers to the frictional heating performed by hydrometeors as they fall relative to the air with a speed equal to their terminal velocity and a force equal to their weight.

Corresponding author address: David M. Romps, Harvard University, 416 Geological Museum, 24 Oxford St., Cambridge, MA 02138.
E-mail: davidromps@gmail.com

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where the second term in (1) has yielded an entropy term from conduction in the bulk (i.e., the interior of the closed volume) plus an entropy source from sensible heating at the surface. Assuming that local temperature gradients are sufficiently small, then the second term is negligible and may be set to zero.[2] Because the lapse rate is dry adiabatic and the prescribed heating Q and surface flux J^z are given, the surface air temperature is all that is needed to evaluate the first and final terms. Eq. (2) then yields a value for the third term, which, when multiplied by a reasonable estimate of the effective dissipation temperature, gives an estimate of the anemonal dissipation. In a steady state, the total kinetic energy is constant, so the kinetic-energy sink $\int_V \mathcal{D}$ equals the kinetic-energy source $-\int_V \mathbf{u} \cdot \boldsymbol{\nabla} p$. In the hydrostatic limit, this equals the buoyancy flux $\int_V \rho w B$, where $B = g(\overline{\rho} - \rho)/\rho$ and the bar denotes a horizontal average. We may then approximate the convective buoyancy fluxes as

$$\int_V \rho w B \approx -T_{\text{diss}} \int_V \left(\frac{Q}{T} - \frac{\mathbf{J} \cdot \boldsymbol{\nabla} T}{T^2} \right) - T_{\text{diss}} \int_S \frac{J^z}{T}, \qquad (3)$$

where T_{diss} is the effective temperature of anemonal dissipation. Therefore, the entropy budget reveals information about the product of vertical velocity and buoyancy and, more generally, about the vigor of convection. Unfortunately, a similar derivation of the anemonal dissipation in a moist atmosphere has proved far more elusive.

Many of the previous studies on entropy in a moist atmosphere have made use of one of two different approximations. Both of these approximations treat the entropy budget of a moist atmosphere as equal to the entropy budget of a dry atmosphere plus an additional heating term corresponding to water vapor. The difference between the two approaches—the "bulk-heating approximation" and the "surface-heating approximation"—is in the different choice of temperature used to divide this external heating to calculate an entropy source. In the bulk-heating approximation, the latent heat of water vapor is counted as "added" to the atmosphere only upon condensation in the bulk, and all other entropy sources from moist processes are assumed to be negligible (Shaw 1923; Lorenz 1967; Peixoto et al. 1991). When viewed this way, the effect of condensation is no different than an external heating source $-L_e e$, where L_e is the latent enthalpy of evaporation and e is the evaporation rate in the bulk, which is negative for condensation. Because this heat is added at the effective condensation temperature T_{cond}, it produces entropy at the rate $-L_e e/T_{\text{cond}}$. This leads to the approximate bulk-heating entropy budget

$$\int_V \left(\frac{Q}{T} - \frac{Le}{T} - \frac{\mathbf{J} \cdot \boldsymbol{\nabla} T}{T^2} + \frac{\mathcal{D}}{T} \right) + \int_S \frac{J^z}{T} = 0. \qquad (4)$$

The task of calculating the anemonal dissipation via Eq. (4) reduces, then, to an estimation of the effective condensation temperature.

In the surface-heating approximation, the latent heat of water vapor is counted as "added" to the atmosphere upon evaporation from the surface, and, again, all other entropy sources from moist processes are assumed to be negligible (Rennó and Ingersoll 1996; Emanuel and Bister 1996; Craig 1996). Therefore, the entropy source $L_e d_v^z/T_{\text{surf}}$ is added to the entropy budget, where d_v^z is the vertical component of the water vapor diffusion flux and T_{surf} is the surface air temperature. This leads to the approximate surface-heating entropy budget

$$\int_V \left(\frac{Q}{T} - \frac{\mathbf{J} \cdot \boldsymbol{\nabla} T}{T^2} + \frac{\mathcal{D}}{T} \right) + \int_S \left(\frac{L_e d_v^z}{T} + \frac{J^z}{T} \right) = 0. \qquad (5)$$

On its face, this approximation is more appealing than the bulk-heating approximation because the dissipation can be obtained from Eq. (5) without needing to estimate the effective condensation temperature.

It is clear that although both approximations may be wrong, they cannot both be right. If both approximations are applied to the same atmosphere in an effort to calculate the anemonal dissipation, then the approximate surface-heating budget will yield an estimate of dissipation that is larger by the amount

$$T_{\text{diss}} L_e d_v^z \left(\frac{1}{T_{\text{cond}}} - \frac{1}{T_{\text{surf}}} \right).$$

Here, use has been made of the fact that $-\int_V e = \int_S d_v^z$ and that L_e does not vary significantly with temperature. Assuming 100 W m^{-2} of latent heat flux, a 300-K surface air temperature, a 280-K effective condensation temperature, and a 280-K effective dissipation temperature, this amounts to a difference of 7 W m^{-2}. This is a large difference when compared to numerical estimates of moist radiative–convective equilibrium (RCE) anemonal dissipation on the order of 1.0 to 1.5 W m^{-2} (Goody 2000; Pauluis et al. 2000) and estimates of global anemonal dissipation in the range of 2 to 10 W m^{-2} (Sverdrup 1917; Brunt 1926; Lettau 1954; Holopainen 1963; Oort 1964; Kung 1966; Peixoto et al. 1991; Boville and Bretherton 2003).

[2] It is safe to ignore the conductive microlayer at the surface, where this assumption does not hold, so long as we set the temperature in the last term equal to the temperature just above the microlayer. This is effectively what happens in numerical models because the heat from the surface is deposited in the first layer, which is much deeper than the microlayer.

In fact, it has been shown that both of these approximations miss large entropy sources. In particular, the surface-heating approximation omits the entropy sources produced by precipitation dissipation, diffusion of water vapor, and nonequilibrium phase changes (Pauluis et al. 2000; Goody 2000; Pauluis and Held 2002a). The bulk-heating approximation omits the entropy produced by precipitation dissipation and a term that is proportional to the condensation rate times the logarithm of total pressure (Pauluis and Held 2002b). These additional terms are the same order of magnitude as the wind-generated dissipation term, so they cannot be omitted from any analysis of the entropy budget that seeks to specify the anemonal dissipation. However, there are two exact budgets that share close similarities to the surface-heating approximation and the bulk-heating approximation: the exact entropy budget and the exact dry-entropy budget, respectively. A proper treatment of the entropy budget was given by Pauluis (2000) and Pauluis and Held (2002a,b). The dry-entropy budget (i.e., the budget for $c_p \log \theta$) was discussed under the name of "potential entropy" in Pauluis (2000). Like the surface-heating approximation, the entropy budget contains a source corresponding to the addition of latent heat at the surface temperature. And, like the bulk-heating approximation, the dry-entropy budget contains a source corresponding to the addition of sensible heat at the condensation temperature.

The papers by Pauluis and Held (2002a,b) demonstrated that the diffusion of water vapor and nonequilibrium phase changes are two of the largest sources in the entropy budget. Furthermore, the numerical simulations presented in those papers showed that the precipitation dissipation is itself much larger than the anemonal dissipation. This led to an explanation for the smallness of anemonal dissipation in a moist atmosphere as compared to a dry atmosphere: the entropy sink produced by surface heating and radiative cooling is relatively fixed and must be balanced by the sum of entropy sources, which are dominated by the diffusion, nonequilibrium phase changes, and precipitation dissipation. This leaves little room for anemonal dissipation. It was also shown by Pauluis and Held (2002b), through a series of approximations, that the diffusion and phase-change terms in the entropy budget may be rewritten, in a steady state, as a sum of terms involving the condensation temperature. This effectively converted their steady-state entropy budget into a steady-state budget of dry entropy, although the connection to dry entropy or "potential entropy" was not explicitly stated there.

In this paper, the emphasis will be on the dry-entropy budget. Section 2 introduces the governing equations, which will be used in both the analytical analysis and the numerical simulations. In contrast to the derivations of Pauluis and Held (2002a,b), these equations include a solid-water phase and realistic heat capacities for the various phases of water. Section 3 introduces the use of integral equations as a way to derive global constraints. This technique is utilized in section 4 to derive the steady-state dry-entropy budget for a moist atmosphere and to demonstrate its connection to the steady-state entropy budget. In section 5, a new cloud-resolving model is introduced and the entropy budgets for various radiative–convective equilibrium simulations are presented. In addition to including ice and realistic heat capacities, these simulations are three-dimensional and use interactive radiation, as compared to the simulations presented by Pauluis and Held (2002a,b), which used two-dimensional domains and Newtonian cooling. Despite these model differences, the values of precipitation and anemonal dissipation are shown to be in good agreement. However, the inclusion of ice introduces a source of dry entropy that, although small, cannot be ignored in a quantitative analysis. Section 6 presents an analysis and discussion of the dry-entropy budget. In this section, the steady-state dry-entropy budget is rewritten to eliminate the precipitation dissipation. The anemonal dissipation is then shown to be the largest of three dry-entropy sources that must balance the sink produced by condensation, sensible heating, and radiation. In addition to the anemonal dissipation, these include an "elevator" term corresponding to the lifting of water and a "conductor" term corresponding to the fuse–melt cycle in the upper troposphere. This formulation of the dry-entropy budget leads to an alternative explanation for the smallness of anemonal dissipation and allows for qualitative predictions of the change in anemonal dissipation under various conditions. Finally, section 7 concludes with a brief summary.

2. Governing equations

In the following equations, we will use an index notation in which summation of repeated indices is implied, Latin indices range over the three spatial dimensions (with index values 1, 2, and 3 corresponding to directions x, y, and z), and Greek letters range over space and time (with index values 0, 1, 2, and 3 corresponding to directions t, x, y, and z). In this notation, the four-vector velocity u^α has components $(u^0, u^1, u^2, u^3) = (1, u, v, w)$. The utility of this definition comes from the fact that the tendency and flux divergence of any specific quantity

$$\frac{\partial}{\partial t}(\rho X) + \nabla \cdot (\rho \mathbf{u} X)$$

can be written as the four-divergence,

$$\partial_\alpha(\rho u^\alpha X).$$

Consider an atmosphere with four components: dry air, water vapor, liquid water, and solid water. Dry air and water vapor will be treated as ideal gases; liquid and solid water will be approximated as phases with zero specific volume. Because the specific volume of condensed water is about a thousand times smaller than that of either air or vapor, this is a good approximation.

Because we will work with mass fractions (mass of component per mass of moist air) instead of mixing ratios (mass of component per mass of dry air), frequent use will be made of an unconventional, but convenient, definition of the parameter ϵ:

$$\epsilon = m_a/m_v - 1,$$

where m_a and m_v are the molar masses of dry air and water, respectively. The traditional definition of ϵ, when working with mixing ratios, is m_v/m_a. It is a mere coincidence that, with the earth's composition of dry air, both of these expressions equal about 0.6. Given this definition of ϵ, the partial pressures of dry air and water vapor are

$$p_a = (1 - q_v - q_l - q_s)\rho R_a T,$$
$$p_v = q_v \rho R_v T, \quad \text{and}$$
$$R_v \equiv (1 + \epsilon) R_a,$$

where q_v is the water-vapor mass fraction, q_l is the liquid-water mass fraction, q_s is the solid-water mass fraction, R_a is the specific gas constant for dry air, and R_v is the specific gas constant for water vapor. The total pressure of a parcel is the sum of these partial pressures,

$$p = R_m \rho T \quad \text{and}$$
$$R_m \equiv (1 + \epsilon q_v - q_l - q_s) R_a,$$

where R_m is the specific gas constant of moist air.

The four components—air, vapor, liquid, and solid—will be given constant specific heats at constant volume: c_{va}, c_{vv}, c_{vl}, and c_{vs}, respectively. The constant-volume specific heat of moist air is

$$c_{vm} = (1 - q_v - q_l - q_s)c_{va} + q_v c_{vv} + q_l c_{vl} + q_s c_{vs},$$

and the related specific heat capacities at constant pressure are

$$c_{pa} = c_{va} + R_a, \quad c_{pv} = c_{vv} + R_v, \quad \text{and} \quad c_{pm} = c_{vm} + R_m,$$

for dry air, water vapor, and moist air, respectively. In previous studies of the entropy budget using cloud-resolving simulations (Pauluis 2000; Pauluis and Held 2002a,b), the values of c_{pv}, c_{vl}, and c_{vs} were set to zero to justify the use of a temperature-independent latent heat. However, setting c_{pv} to zero implies that the constant-volume heat capacity of water vapor is negative, which means that water vapor at constant volume will warm up as it loses energy and get colder as heat is added. This unphysical peculiarity is avoided here by using a numerical model with realistic values of the heat capacities.

Three diffusion fluxes will be used, each of which has units of kg m^{-2} s^{-1}. These are the diffusion fluxes of water vapor $\mathbf{d}_v$, liquid water $\mathbf{d}_l$, and solid water $\mathbf{d}_s$. For precipitation, $\mathbf{d}_l$ and $\mathbf{d}_s$ encompass the flux associated with the hydrometeors' terminal velocities.

One approximation that will be used is to neglect the momentum and kinetic energy associated with the diffusion fluxes of water. In particular, the velocities of water vapor, liquid water, and solid water are

$$\mathbf{v}_v = \mathbf{u} + \frac{\mathbf{d}_v}{q_v \rho},$$
$$\mathbf{v}_l = \mathbf{u} + \frac{\mathbf{d}_l}{q_l \rho}, \quad \text{and}$$
$$\mathbf{v}_s = \mathbf{u} + \frac{\mathbf{d}_s}{q_s \rho},$$

so the true specific momenta are $\mathbf{v}_v$, $\mathbf{v}_l$, and $\mathbf{v}_s$, and the true specific kinetic energies are $v_v^2/2$, $v_l^2/2$, and $v_s^2/2$. However, the equations are greatly simplified by treating the specific momentum of all water as $\mathbf{u}$. With this approximation, water vapor advects momentum density $q_v \rho \mathbf{u}$ with velocity $\mathbf{v}_v$, and similarly for liquid and solid water. In addition, the kinetic-energy density of a moist parcel is $\rho u^2/2$ regardless of any water content.

Note that the neglect of a hydrometeor's free-fall kinetic energy is well justified when compared to the magnitudes of other terms in the energy and entropy budgets. The kinetic energy exported out of the atmosphere by hydrometeors falling through the lower boundary can be estimated by assuming 100 W m^{-2} of evaporation at the surface, which corresponds to 4×10^{-5} kg m^{-2} of water vapor, and a terminal velocity of 10 m s^{-1}. The resulting export of kinetic energy in a steady state is

$$(4 \times 10^{-5}) \times \frac{1}{2} \times 10^2 \approx 2 \times 10^{-3} \text{W m}^{-2}.$$

This is a negligible power density compared to the processes of interest to us. For example, given an effective fall height of 10^4 meters, precipitation dissipation is

$$(4 \times 10^{-5}) \times 9.8 \times 10^4 \approx 4 \text{ W m}^{-2}.$$

Because the wind-generated dissipation is also expected to be of this order of magnitude, the export of kinetic energy by precipitation will not enter the dominant balance that sets the anemonal dissipation.

The governing equations are

$$
\begin{aligned}
\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha] &= 0 \\
\partial_\alpha(q_v \rho v_v^\alpha) &= e \\
\partial_\alpha(q_l \rho v_l^\alpha) &= -e + m \\
\partial_\alpha(q_s \rho v_s^\alpha) &= -m \\
\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha \mathbf{u}] + \partial_\alpha(q_v \rho v_v^\alpha \mathbf{u}) + \partial_\alpha(q_l \rho v_l^\alpha \mathbf{u}) + \partial_\alpha(q_s \rho v_s^\alpha \mathbf{u}) &= \rho \mathbf{g} - \nabla p - \nabla \cdot \tau \\
\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha E_a^{\text{tot}}] + \partial_\alpha(q_v \rho v_v^\alpha E_v^{\text{tot}}) + \partial_\alpha(q_l \rho v_l^\alpha E_l^{\text{tot}}) + \partial_\alpha(q_s \rho v_s^\alpha E_s^{\text{tot}}) &= Q - \nabla \cdot (p_a \mathbf{u}) - \nabla \cdot (p_v \mathbf{v}_v) \\
&\quad - \nabla \cdot (\mathbf{u} \cdot \tau) - \nabla \cdot \mathbf{J}.
\end{aligned}
$$

Here, e is the evaporation rate in the bulk with dimensions of mass per volume per time; negative values of e correspond to condensation. Similarly, m is the melting rate in the bulk. The viscous stress tensor is represented by τ, $\mathbf{J}$ is the conductive flux of sensible heat, and the variable Q is the radiative heating per volume. The first four equations are the continuity equations for dry air, water vapor, liquid water, and solid water, respectively. In the momentum equation, water transports only the momentum corresponding to the dry-air velocity, as discussed above. The final equation is the budget for total energy, where the specific total energies for dry air, water vapor, and liquid water are

$$
\begin{aligned}
E_a^{\text{tot}} &= c_{va}(T - T_{\text{trip}}) + \frac{1}{2}u^2 + \phi \\
E_v^{\text{tot}} &= c_{vv}(T - T_{\text{trip}}) + \frac{1}{2}u^2 + \phi + E_{0v} \\
E_l^{\text{tot}} &= c_{vl}(T - T_{\text{trip}}) + \frac{1}{2}u^2 + \phi \\
E_s^{\text{tot}} &= c_{vs}(T - T_{\text{trip}}) + \frac{1}{2}u^2 + \phi - E_{0s}.
\end{aligned}
$$

Here, $\phi = gz$ is the specific gravitational potential energy, $T_{\text{trip}} = 273.16$ K is the triple-point temperature, E_{0v} is the specific internal energy of water vapor at the triple point, and $-E_{0s}$ is the specific internal energy of ice at the triple point. The governing equations can be rearranged to give

$$\partial_\alpha(\rho u^\alpha) = -\nabla \cdot (\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s), \tag{6}$$

$$\partial_\alpha(q_v \rho u^\alpha) = e - \nabla \cdot \mathbf{d}_v, \tag{7}$$

$$\partial_\alpha(q_l \rho u^\alpha) = -e + m - \nabla \cdot \mathbf{d}_l, \tag{8}$$

$$\partial_\alpha(q_s \rho u^\alpha) = -m - \nabla \cdot \mathbf{d}_s, \tag{9}$$

$$\partial_\alpha(\rho u^\alpha \mathbf{u}) = \rho \mathbf{g} - \nabla p - \nabla \cdot \tau - \partial_i[(d_v^i + d_l^i + d_s^i)\mathbf{u}], \quad \text{and} \tag{10}$$

$$
\begin{aligned}
\partial_\alpha(\rho u^\alpha E^{\text{tot}}) &= Q - \nabla \cdot (p\mathbf{u}) - \nabla \cdot (\mathbf{u} \cdot \tau) - \nabla \cdot \mathbf{J} \\
&\quad - \nabla \cdot [(E_v^{\text{tot}} + R_v T)\mathbf{d}_v] - \nabla \cdot (E_l^{\text{tot}} \mathbf{d}_l) \\
&\quad - \nabla \cdot (E_s^{\text{tot}} \mathbf{d}_s),
\end{aligned} \tag{11}
$$

where $E^{\text{tot}} = c_{vm}(T - T_{\text{trip}}) + ½u^2 + \phi + q_v E_{0v} - q_s E_{0s}$.

3. Integral equations

Before we discuss the integral-equation approach to entropy, it is useful to recall the standard method for deriving the entropy budget. The specific entropies of dry air, water vapor, liquid water, and solid water are

$$s_a = c_{pa} \log(T/T_{\text{trip}}) - R_a \log(p_a/p_{\text{trip}}), \tag{12}$$

$$s_v = c_{pv} \log(T/T_{\text{trip}}) - R_v \log(p_v/p_{\text{trip}}) + s_{0v}, \tag{13}$$

$$s_l = c_{vl} \log(T/T_{\text{trip}}), \quad \text{and} \tag{14}$$

$$s_s = c_{vs} \log(T/T_{\text{trip}}) - s_{0s}, \tag{15}$$

where $s_{0v} = E_{0v}/T_{\text{trip}} + R_v$, $s_{0s} = E_{0s}/T_{\text{trip}}$, and $p_{\text{trip}} = 611.65$ Pa is the triple-point pressure. Using the four-vector notation, the differential entropy equation can then be written as $\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha s_a] + \partial_\alpha(q_v \rho v_v^\alpha s_v) + \partial_\alpha(q_l \rho v_l^\alpha s_l) + \partial_\alpha(q_s \rho v_s^\alpha s_s) =$ Entropy sources. In a steady state with $\mathbf{u} = 0$ at the boundaries and diffusion fluxes zero at all boundaries except the lower boundary, integrating over a closed volume yields

$$\int_V \text{Entropy sources} + \int_S (\mathbf{d}_v^z s_v + \mathbf{d}_l^z s_l + \mathbf{d}_s^z s_s) = 0. \tag{16}$$

The standard approach is to catalog all the local entropy sources and sinks and insert them into the above equation to get the entropy budget.

There are a couple important observations to make about the entropy budget. First, the entropy budget does not contain any new information that is not already contained within the differential governing equations and the boundary conditions. This fact is sometimes obscured by the standard approach of cataloging entropy sources and sinks without direct reference to the governing equations. Second, the entropy budget in Eq. (16) is a global statement about steady-state solutions, so it contains far less information than the differential equations. On the other hand, there are an infinite number of integral equations that can be derived, the full set of which contains the same information as the differential governing equations and boundary conditions.

An alternative approach to cataloging entropy sources is to use the governing equations directly. In this method, we begin with any specific quantity, multiply by ρu^α, take the space–time divergence ∂_α, and integrate over space and time. Assuming that the dry-air velocity $\mathbf{u}$ is zero on the spatial boundaries and a true steady state has been achieved, then this integral is guaranteed to be zero. The next step is to remove derivatives from the integrand using the differential governing equations. The resulting equation is the global relation corresponding to the chosen specific quantity.

As a simple example, consider the specific kinetic energy $u^2/2$. From the governing equations, we can derive the differential equation for kinetic energy,

$$\partial_\alpha\left(\rho u^\alpha \frac{1}{2}u^2\right) = -\nabla\cdot\left[\frac{1}{2}u^2(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)\right] + \rho\mathbf{g}\cdot\mathbf{u} - \mathbf{u}\cdot\nabla p + \mathbf{u}\cdot\mathcal{F}, \qquad (17)$$

where $\mathcal{F} \equiv -\nabla\cdot\tau$ is the frictional force per volume. Taking the integral over space and time yields

$$\int_V (g\rho w + \mathbf{u}\cdot\nabla p + \mathcal{D}) = 0,$$

where $\mathcal{D} \equiv -\tau\colon \nabla\mathbf{u}$ is the anemonal dissipation per volume. In a dry atmosphere, the integral of ρw is zero because there is no net advective flux of mass in the vertical. In the moist case, this same argument cannot be made because of the diffusion and terminal-velocity fluxes of water. Instead, we can make two separate statements to the effect that there is no net vertical flux of dry air and no net vertical flux of water in a steady state:

$$\int_V (1 - q_v - q_l - q_s)\rho w = 0 \quad \text{and}$$

$$\int_V (q_v + q_l + q_s)\rho w + \int_V (d_v^z + d_l^z + d_s^z) = 0.$$

Adding these equations, we find that

$$\int_V \rho w = -\int_V (d_v^z + d_l^z + d_s^z).$$

Therefore, the integral of the kinetic-energy equation can be written as

$$-\int_V \mathbf{u}\cdot\nabla p = \int_V [\mathcal{D} + \mathbf{g}\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)]. \qquad (18)$$

In an atmosphere with a negligible net vertical diffusion of water vapor, this states that the pressure work equals the sum of anemonal dissipation and precipitation dissipation.

4. The dry-entropy budget

The technique outlined in the previous section can be used to find the entropy budget directly from the differential equations. Naturally, the relevant specific quantity is the specific entropy of moist air. This can be derived from Eqs. (12)–(14) as

$$\begin{aligned}
s &= (1 - q_v - q_l - q_s)s_a + q_v s_v + q_l s_l + q_s s_s \\
&= c_{pm}\log(T/T_0) && \text{I}\\
&\quad - R_m \log(p/p_0) && \text{II}\\
&\quad + R_a(1 + \epsilon q_v - q_l - q_s)\log(1 + \epsilon q_v - q_l - q_s) && \text{III}\\
&\quad - R_a(1+\epsilon)q_v \log[(1+\epsilon)q_v] && \text{IV}\\
&\quad - R_a(1 - q_v - q_l - q_s)\log(1 - q_v - q_l - q_s) && \text{V}\\
&\quad + q_v s_{0v} && \text{VI}\\
&\quad - q_s s_{0s}. && \text{VII}
\end{aligned} \qquad (19)$$

Although this entire expression for s could be used to derive an integral equation, this is not necessary. Instead, we will focus on the dry-entropy budget, which follows directly from the sum of terms I and II. Note that the sum of terms I and II is the specific entropy of a dry gas with the same temperature, pressure, heat capacity, and gas constant as the moist air: thus the term "dry entropy." Other integral equations may be derived from the remaining terms. As we will soon see, the integral equation associated with the sum of terms III and IV implies a constraint on the effective pressure of condensation. Terms V, VI, and VII can be used independently to find integral equations that are a subset of those derived in appendix B.

Integrating the four-divergence of ρu^α times the specific dry entropy (the sum terms I and II) gives

$$\int_V \partial_\alpha\{\rho u^\alpha[c_{pm}\log(T/T_0) - R_m\log(p/p_0)]\} = 0. \quad (20)$$

We will leave the values of p_0 and T_0 unspecified for the time being; a convenient choice of values will make itself manifest shortly. As shown in appendix A, the integrand in Eq. (20) can be rewritten with help from the governing equations to yield

$$\int_V \left[(c_{pv} - c_{vl})e\log\left(\frac{T}{T_0}\right) + (c_{vl} - c_{vs})m\log\left(\frac{T}{T_0}\right) - R_v(e - \nabla\cdot\mathbf{d}_v)\log\left(\frac{p}{p_0}\right) + \frac{Q}{T} - \frac{L_e e}{T} - \frac{L_m m}{T} - \frac{\nabla\cdot\mathbf{J}}{T} + \frac{\mathcal{D}}{T} + \frac{\mathbf{g}\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)}{T}\right] + \int_S (c_{pv}d_v^z + c_{vl}d_l^z + c_{vs}d_s^z)\log(T/T_0) = 0, \quad (21)$$

where

$$L_e = E_{0v} + R_vT + (c_{vv} - c_{vl})(T - T_{\rm trip}), \quad (22)$$

$$L_m = E_{0s} + (c_{vl} - c_{vs})(T - T_{\rm trip}). \quad (23)$$

This is the steady-state dry-entropy budget. The derivation given here is similar to the method used in Pauluis (2000), but this budget differs by the presence of the two melting terms and the diffusion of water up a geopotential gradient. In simulations with subgrid diffusion of water vapor, this latter term cannot be neglected. This budget also differs from the steady-state form of the budget studied by Pauluis and Held (2002a,b) because it includes the terms proportional to the difference in specific heat capacities, the term corresponding to the latent heat of melting, and the term corresponding to the diffusion of water vapor up the geopotential gradient.

Recall that T_0 and p_0 are arbitrary. It is particularly convenient to choose p_0 and T_0 to be the average pressure and temperature at the surface. Neglecting variations of surface temperature from the average value, the surface integral is zero. By neglecting variations of surface pressure from the average value and assuming that the diffusion flux of water vapor $\mathbf{d}_v$ is nonnegligible only very close to the surface, the remaining terms involving $\mathbf{d}_v$ are zero as well. This leads to an approximation of the dry-entropy budget

$$\int_V \left[(c_{pv} - c_{vl})e\log\left(\frac{T}{T_0}\right) + (c_{vl} - c_{vs})m\log\left(\frac{T}{T_0}\right) - R_v e\log\left(\frac{p}{p_0}\right) + \frac{Q}{T} - \frac{L_e e}{T} - \frac{L_m m}{T} - \frac{\nabla\cdot\mathbf{J}}{T} + \frac{\mathcal{D}}{T} + \frac{\mathbf{g}\cdot(\mathbf{d}_l + \mathbf{d}_s)}{T}\right] = 0 \quad (24)$$

that is valid for p_0 and T_0 equal to the surface values and for $\mathbf{d}_v$ nonzero only near the surface.

It is possible to demonstrate the connection between the steady-state dry-entropy budget in Eq. (21) to the steady-state entropy budget by using the integral equation associated with the sum of terms III and IV in Eq. (19). As shown in appendix B, they give us the following relation:

$$\int_V R_v(e - \nabla\cdot\mathbf{d}_v)\log\left(\frac{p_v}{p}\right) = 0. \quad (25)$$

When subtracted from (21), it converts the logarithm of total pressure to a logarithm of the partial pressure of water vapor. Now, let us use the freedom in T_0 and p_0 to choose $T_0 = T_{\rm trip}$ and $p_0 = p_{\rm trip}$. Noting that

$$-\frac{L_e e}{T} = -\frac{e}{T}[E_{0v} + R_vT + (c_{vv} - c_{vl})(T - T_{\rm trip})]$$
$$= R_v e\log\left\{\exp\left[\frac{E_{0v} - (c_{vv} - c_{vl})T_{\rm trip}}{R_v}\left(\frac{1}{T_{\rm trip}} - \frac{1}{T}\right)\right]\right\} - es_{0v},$$

we may rewrite the terms involving e as

$$\int_V \left[(c_{pv} - c_{vl})e \log\left(\frac{T}{T_{\rm trip}}\right) - R_v(e - \nabla \cdot \mathbf{d}) \log\left(\frac{p_v}{p_{\rm trip}}\right) - \frac{L_e e}{T} \right]$$

$$= \int_V \left(R_v e \log\left\{ \frac{p_{\rm trip}}{p_v} \left(\frac{T}{T_{\rm trip}}\right)^{\frac{c_{pv}-c_{vl}}{R_v}} \exp\left[\frac{E_{0v} - (c_{vv} - c_{vl})T_{\rm trip}}{R_v} \left(\frac{1}{T_{\rm trip}} - \frac{1}{T}\right)\right]\right\} \right.$$

$$\left. + R_v \nabla \cdot \left[\mathbf{d}_v \log\left(\frac{p_v}{p_{\rm trip}}\right)\right] - R_v \mathbf{d}_v \cdot \nabla \log p_v - e s_{0v} \right)$$

$$= \int_V \left[-R_v e \log\left(\frac{p_v}{p_v^{*,l}}\right) - R_v \mathbf{d}_v \cdot \nabla \log p_v \right] + \int_S d_v^z \left[-R_v \log\left(\frac{p_v}{p_{\rm trip}}\right) + s_{0v}\right].$$

In the last line, we have used the fact that the saturation vapor pressure with respect to liquid water is

$$p_v^{*,l} = p_{\rm trip}\left(\frac{T}{T_{\rm trip}}\right)^{\frac{c_{pv}-c_{vl}}{R_v}} \exp\left[\frac{E_{0v} - (c_{vv} - c_{vl})T_{\rm trip}}{R_v}\left(\frac{1}{T_{\rm trip}} - \frac{1}{T}\right)\right].$$

Similarly, it is straightforward to show that the terms involving m may be written as

$$\int_V \left[(c_{vl} - c_{vs})m \log\left(\frac{T}{T_{\rm trip}}\right) - \frac{L_m m}{T}\right] = \int_V R_v m \log\left(\frac{p_v^{*,s}}{p_v^{*,l}}\right) - \int_S d_s^z s_{0s},$$

where

$$p_v^{*,s} = p_{\rm trip}\left(\frac{T}{T_{\rm trip}}\right)^{\frac{c_{pv}-c_{vs}}{R_v}} \exp\left[\frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\rm trip}}{R_v}\left(\frac{1}{T_{\rm trip}} - \frac{1}{T}\right)\right]$$

is the saturation vapor pressure with respect to solid water. Substituting these expressions into the dry-entropy budget results in

$$\int_V \left[\frac{Q}{T} + \frac{\mathcal{D}}{T} + \frac{\mathbf{g} \cdot (\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)}{T} - \frac{\nabla \cdot \mathbf{J}}{T} - R_v e \log\left(\frac{p_v}{p_v^{*,l}}\right) - R_v m \log\left(\frac{p_v^{*,l}}{p_v^{*,s}}\right) - R_v \mathbf{d}_v \cdot \nabla \log p_v \right]$$

$$+ \int_S (s_v d_v^z + s_l d_l^z + s_s d_s^z) = 0. \tag{26}$$

This is the steady-state entropy budget. Because the steady-state dry-entropy budget in Eq. (21) is valid for any choice of p_0 and T_0, and because p_0 and T_0 were restricted to lie on the phase boundary in the process of deriving Eq. (26), the steady-state entropy budget in Eq. (26) may be considered a special case of (21). By restricting p_0 and T_0 to the phase boundary, we have recovered the familiar terms corresponding to nonequilibrium phase changes and the diffusion of water vapor down a partial-pressure gradient. This budget differs from the one studied by Pauluis and Held (2002a,b) by the inclusion of the melting term and the term corresponding to the diffusion of water vapor up the geopotential gradient. It also differs in the implicit definitions of the saturation vapor pressures, which depend on the difference in specific heat capacities between the respective phases.

5. Numerical results

The model used to investigate the implications of Eq. (21) is Das Atmosphärische Modell (DAM), which was built by the author with the specific goal of faithfully integrating the governing equations laid out in section 2. DAM is a three-dimensional, finite-volume, fully compressible, nonhydrostatic, cloud-resolving model

TABLE 1. The energy budget (W m^{-2}), without (w/o diff) and with (w/diff) diffusion of heat and water.

	5, 300	5, 303	10, 300	10, 303	5, 300	5, 303	10, 300	10, 303
	w/o diff	w/o diff	w/o diff	w/o diff	w/ diff	w/ diff	w/ diff	w/ diff
Q	−110.00	−130.59	−118.62	−139.69	−108.71	−129.30	−116.97	−137.88
J^z_{surf}	7.42	9.67	3.25	5.13	7.20	9.44	3.19	5.05
$h_v d^z_v$	107.00	126.76	120.59	141.37	105.96	125.76	118.98	139.55
$h_l d^z_l$	−4.49	−5.88	−5.24	−6.81	−4.45	−5.84	−5.17	−6.72
Sum	−0.08	−0.04	−0.03	0.00	0.00	0.06	0.04	−0.01

with prognostic momentum, virtual potential temperature, total mass, and masses of each of the water components. DAM uses height as the vertical coordinate in an Arakawa C-type grid with uniform horizontal spacing and variable vertical spacing. The upper and lower boundary conditions are that of a no-slip, rigid lid, and the horizontal domain is doubly periodic.

Advection is performed in a conservative-flux form. DAM may be run with any advection order subject to the condition that the stencil not extend past adjacent subdomains. For the simulations presented in the next section, third-order advection is used. A positive-definite flux correction is used for the water densities following Smolarkiewicz (1989). To deal with acoustic modes and fast gravity waves, DAM uses a conservative split-time scheme whereby the terms responsible for the acoustic and gravity modes are integrated explicitly with a small time step (Klemp et al. 2007). In the small time step, terms with horizontal derivatives are integrated with forward–backward differencing. Because the vertical grid spacing is typically much smaller than the horizontal grid spacing, terms responsible for vertical sound waves are treated implicitly. The large time steps are integrated in time using a second-order Runge–Kutta scheme.

Microphysics is treated with the Lin–Lord–Krueger six-class microphysics scheme (Lin et al. 1983; Lord et al. 1984; Krueger et al. 1995). The six classes are water vapor, cloud liquid, cloud ice, rain, snow, and graupel. Interactive radiation was used with the shortwave flux set equal to the diurnally averaged radiance at the equator on the first of January. The interactive radiation scheme is from the National Center for Atmospheric Research (NCAR) Community Climate Model (CCM; Kiehl et al. 1998). A Smagorinsky model is used to determine the eddy diffusivity. Surface fluxes are determined by a simple bulk parameterization in which the near-surface velocity is replaced by a fixed background velocity. This background velocity has no effect on the dynamics other than through the parameterization of surface fluxes.

Eight simulations of RCE over the ocean were performed with DAM to investigate the dry-entropy budget. The two parameters that vary among the runs are the background wind speed used in the bulk surface-flux parameterization (5 and 10 m s^{-1}) and the sea surface temperature (300 and 303 K). In four of the runs, the eddy viscosity was used to diffuse momentum, but there was no subgrid diffusion of sensible heat or water. The other set of four runs was identical except the eddy diffusivity was also used to model subgrid diffusion of heat and water. The simulations were run at a 2-km horizontal resolution on a doubly periodic, horizontal domain with 32 km per side. This is a relatively small domain, but previous studies have found no difference in the statistics of radiative–convective equilibrium on a 60-km-wide domain versus a 120-km-wide domain (Robe and Emanuel 1996; Tompkins 2000). The 64 vertical grid points were smoothly spaced from just under 100 m near the surface, to a constant 500 m through most of the vertical range, to just over 1000 m near the top, for a total of 32 km in the vertical.

Each simulation was run to a steady state, which was confirmed by a zero rate of change in total energy. The simulations were then run for 8 months of model time, during which statistics were collected. In each case, the majority of the clouds detrain at or below the melting line at a height of about 5 or 6 km. A smaller number of deep convective plumes reach up to a maximum height between 15 and 17 km, depending on the sea surface temperature and background wind speed. Starting as updrafts in the subcloud layer, the fastest deep convective plumes accelerate from velocities around 10 m s^{-1} at the freezing line to velocities between 20 and 25 m s^{-1} at a height between 10 and 15 km.

In sections 3 and 4, integral equations were derived by taking averages over infinite time. In practice, the integral equations will be valid so long as the terms are averaged over a sufficiently long period of a steady state. The time period of 8 months was chosen to err on the side of caution. The energy budgets over this time period are shown in Table 1. The columns are labeled by the background wind speed used for the surface-flux parameterization (5 or 10 m s^{-1}) and the sea surface temperature (300 or 303 K). The first four columns are from the simulations without any subgrid diffusion of

TABLE 2. The dry-entropy budget (mW m^{-2} K^{-1}).

	5, 300	5, 303	10, 300	10, 303	5, 300	5, 303	10, 300	10, 303
	w/o diff	w/o diff	w/o diff	w/o diff	w/ diff	w/ diff	w/ diff	w/ diff
Q/T	−402.6	−473.1	−432.2	−503.7	−397.9	−468.4	−426.4	−497.6
$-L_e e/T$	375.0	438.6	418.5	484.8	373.7	437.8	415.7	481.7
$-\nabla \cdot \mathbf{J}/T$	24.8	32.0	10.9	17.0	22.2	29.3	8.6	14.5
$\mathbf{g} \cdot (\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)/T$	11.5	13.9	12.6	15.3	11.1	13.4	12.1	14.6
$-R_v(e - \nabla \cdot \mathbf{d}_v) \log(p/p_0)$	−8.5	−10.7	−9.4	−11.9	−7.9	−10.0	−8.7	−11.0
$(c_{pv} - c_{vl})e \log(T/T_0)$	−7.1	−7.5	−7.3	−7.5	−7.6	−8.0	−7.7	−8.0
$\mathcal{D}/T$	5.3	6.0	5.8	6.5	4.6	5.2	5.0	5.6
$(c_{vl} - c_{vs})m \log(T/T_0)$	2.9	3.0	3.0	2.9	3.2	3.2	3.2	3.2
$-L_m m/T$	−1.3	−1.3	−1.3	−1.3	−1.4	−1.4	−1.4	−1.4
$(c_{pv}d_v^z + c_{vl}d_l^z + c_{vs}d_s^z)\log(T/T_0)$	0.7	−0.3	0.4	−0.8	0.7	−0.3	0.4	−0.8
Sum	0.7	0.7	0.9	1.2	0.6	0.9	0.8	0.8

sensible heat and water; the last four columns report results from simulations that do use subgrid diffusion of scalars. The average rate of change in the total energy of the atmosphere over this period for each of the simulations is less than 0.1 W m^{-2}. When integrated over several 1-month periods, the simulations exhibit a root-mean-square change in total energy of about 0.4 W m^{-2}, which suggests that a smaller time integration might have been sufficient considering that the term of interest—the anemonal dissipation—is a few times larger than this.

The dry-entropy budgets for the eight runs are shown in Table 2 in units of mW m^{-2} K^{-1}. The values of p_0 and T_0 that are used in calculating the table are the mean surface pressure and mean surface air temperature for each of the runs. The values are arranged in order of decreasing magnitude, from the large radiative sink to the small source that is proportional to $\log(T/T_0)$ at the surface. For a sufficiently long average over time, the sum of the terms in Table 2 should be zero. However, the sums range from 0.6 to 1.2 mW m^{-2} K^{-1}, indicating that there is some small bias remaining even after an 8-month integration. This may reflect the fact that some aspects of the RCE are still not in equilibrium even after 8 months. For example, water vapor in the stratosphere had not yet equilibrated by the end of the 8 months, and there are some indications that the radiative forcing in parts of stratosphere had not yet reached a steady value. Fortunately, the summation errors in the dry-entropy budgets are small compared to the sources from anemonal dissipation, which take values between 4.6 and 6.5 mW m^{-2} K^{-1}.

6. Analysis and discussion

It merits a moment to consider the physical interpretation of each of the terms in Eq. (21) and Table 2. The three largest terms correspond to the heat sources from radiation, condensation, and sensible heat, respectively. The $\mathcal{D}/T$ term comes from the addition of heat by the dissipation of wind shear by viscosity. Of course, in a numerical simulation, the dissipation that is measured is really the transport of kinetic energy down to subgrid scales. As long as the grid scale is within the inertial subrange, which is the case in large-eddy simulations, this power will equal the dissipation by viscosity. With currently available computing power, and with the requirement that the simulations be run to a steady state, the simulations presented here qualify as very-large-eddy simulations; in other words, the grid scale is not within the inertial subrange. This means that the 2-km resolution used here may not be small enough for the dry-entropy budget to have achieved convergence, although this same caveat applies to any study using cloud-resolving models with grid spacing greater than roughly 250 m (Bryan et al. 2003).

The term proportional to the gravitational acceleration times the vertical **d** fluxes is the precipitation dissipation term, corresponding to the heating produced by the conversion of gravitational potential energy into heat via friction as the hydrometeors fall. Recall that the **d** vectors include diffusive fluxes in addition to the hydrometeor terminal velocities. In the simulations without diffusion of water, $\mathbf{d}_v$ is zero except at the surface, and $\mathbf{d}_l$ and $\mathbf{d}_s$ are equal to the terminal-velocity fluxes. In those simulations, the $\mathbf{d}_v$ term may be neglected. The term involving the logarithm of pressure is closely related to the precipitation dissipation; its meaning will be discussed shortly.

There are three terms that are proportional to the specific heat capacities of water. These terms correspond to the addition or removal of heat by water as it moves up and down the atmosphere's temperature gradient. For example, liquid precipitation that is falling toward the ground is moving up a temperature gradient. In the governing equations, it is assumed that the precipitation immediately adjusts to the temperature of the environment, so the precipitation must absorb sensible heat as it falls. As the temperature of the precipitation changes by dT, the atmosphere experiences a heat source equal to $-c_{vl}dT$. Therefore, the dry-entropy source produced by a unit mass of liquid precipitation that initially forms at temperature T and exits through the lower boundary at temperature T_0 is

$$\int_T^{T_0} \frac{-d(c_{vl}T')}{T'} = c_{vl}\log(T/T_0).$$

In general, the sources and sinks for liquid water are $m - e$ in the bulk and d_l^z at the lower boundary. This leads to a net dry-entropy source of

$$\int_V (m - e)c_{vl}\log(T/T_0) + \int_S d_l^z c_{vl}\log(T/T_0).$$

Similar expressions apply to water vapor and solid water, the sum of which yields the three terms involving $\log(T/T_0)$ in Eq. (21) and Table 2.

Although the net melting of water, $\int_V m$, is zero, the term $-\int_V L_m m/T$ is not. This stems from the fact that liquid water often becomes supercooled in updrafts that pass the melting line. The temperature at which liquid water fuses is typically many degrees colder than the temperature at which it melts. This suggests that there should be a net transport of heat from the relatively warm melting line (where water melts and absorbs latent heat) to the relatively cool upper troposphere (where water fuses and releases latent heat), which would be a source of dry entropy. Indeed, this process does move heat from warm to cold, and yet the numerical values in Table 2 for $-\int_V L_m m/T$ are negative, indicating a sink. This apparent incongruity is resolved by noting that the temperature dependence of L_m is greater than that of $1/T$. Using the expression for L_m in Eq. (23), and using the fact that $\int_V m = 0$, we see that

$$\int_V \left(-\frac{L_m m}{T}\right) = [(c_{vl} - c_{vs})T_{\text{trip}} - E_{0s}]\int_V \frac{m}{T}.$$

Because m is positive at high T and negative at low T, the integral on the right is negative; because c_{vl} is so much larger than c_{vs}, the term outside the integral is positive. Therefore, the overall term is negative, in agreement with the values in Table 2. Although this term is a sink, the $(c_{vl} - c_{vs})m\log(T/T_0)$ term is a source of greater magnitude, confirming our intuition that the fuse/melt cycle should provide a net source of dry entropy.

To gain some additional insight, it is useful to employ effective temperatures where possible. For example, the effective temperature of condensation may be defined as

$$T_c \equiv \frac{\int_V L_e e}{\int_V L_e e/T},$$

which guarantees that the condensation source of dry entropy may be written as $(-\int_V L_e e)/T_c$. In this way, effective temperatures can also be defined for radiation, sensible heating, anemonal dissipation, and precipitation dissipation. Table 3 shows these terms in this disaggregated fashion. We can see that the anemonal dissipation ranges from 1.3 to 1.8 W m^{-2} and the precipitation dissipation ranges from 3.0 to 4.2 W m^{-2}. These values are in excellent agreement with the three-dimensional, cloud-resolving simulation of Pauluis et al. (2000), in which values of 1.4 and 3.6 W m^{-2} were obtained for the anemonal and precipitation dissipation, respectively. On the other hand, the simulations of Pauluis and Held (2002a) found that the anemonal and precipitation dissipation were 1.0 and 3.7 W m^{-2}, respectively. That value of the anemonal dissipation is outside the range found here, which may be explained by details of their model. Unlike the model used here and in Pauluis et al. (2000), the model used by Pauluis and Held (2002a) was two-dimensional and used Newtonian cooling instead of interactive radiation.

To tease out the effects of subgrid diffusion models on the dry-entropy budget, four of the runs did not use any subgrid diffusion of water or heat, whereas the other four did. Comparing the simulations with and without the diffusion of these scalars in Table 2, it is clear that the largest changes in dry-entropy sources and sinks occur for the sources and sinks with the largest overall magnitudes: radiation, condensation, and sensible heating. In each of the four pairs, the addition of scalar diffusion leads to a decrease in the radiation sink that more than compensates for the decrease in the condensation and sensible heating sources. Therefore, scalar diffusion causes a reduction in the net sink from radiation, condensation, and sensible heating. It is clear from Table 3 that the decrease in the radiation sink with the addition of scalar diffusion stems not from a change in effective temperature, but from a decrease in the total energy flux. The simulations with scalar diffusion are slightly warmer and moister throughout most of the troposphere and at the surface, where surface fluxes are subsequently suppressed.

With scalar diffusion, the dry-entropy source from sensible heating is reduced for all four pairs of runs. As seen from Table 3, this is only partly explained by a

TABLE 3. The dry-entropy budget with the sources and sinks written as an energy flux (W m^{-2}) over an effective temperature (K), where applicable.

	5, 300	5, 303	10, 300	10, 303	5, 300	5, 303	10, 300	10, 303
	w/o diff	w/o diff	w/o diff	w/o diff	w/ diff	w/ diff	w/ diff	w/ diff
Q/T	−110.0/273.3	−130.6/276.1	−118.6/274.5	−139.7/277.3	−108.7/273.2	−129.3/276.0	−117.0/274.3	−137.9/277.1
$-L_e e/T$	104.4/278.3	123.1/280.6	117.3/280.3	136.9/282.5	103.4/276.8	122.2/279.2	115.9/278.8	135.3/281.0
$-\nabla \cdot \mathbf{J}/T$	7.4/299.1	9.7/301.8	3.3/299.7	5.1/302.6	7.2/324.1	9.4/322.1	3.2/369.6	5.0/347.8
$\mathbf{g} \cdot (\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)/T$	3.2/274.2	3.8/276.7	3.5/275.5	4.2/278.1	3.0/273.7	3.7/276.2	3.3/275.0	4.1/277.7
$-R_v(e - \nabla \cdot \mathbf{d}_v) \log(p/p_0)$	−0.0085	−0.0107	−0.0094	−0.0119	−0.0079	−0.0100	−0.0087	−0.0110
$(c_{pv} - c_{vl})e \log(T/T_0)$	−0.0071	−0.0075	−0.0073	−0.0075	−0.0076	−0.0080	−0.0077	−0.0080
$\mathcal{D}/T$	1.5/276.5	1.7/278.3	1.6/276.0	1.8/278.4	1.3/275.8	1.5/277.6	1.4/274.9	1.6/277.2
$(c_{vl} - c_{vs})m \log(T/T_0)$	0.0029	0.0030	0.0030	0.0029	0.0032	0.0032	0.0032	0.0032
$-L_m m/T$	−0.0013	−0.0013	−0.0013	−0.0013	−0.0014	−0.0014	−0.0014	−0.0014
$(c_{pv}d_v^z + c_{vl}d_l^z + c_{vs}d_s^z) \log(T/T_0)$	0.0007	−0.0003	0.0004	−0.0008	0.0007	−0.0003	0.0004	−0.0008
Sum	0.0007	0.0007	0.0009	0.0012	0.0006	0.0009	0.0008	0.0008

reduced sensible heat flux from the surface. In fact, the larger contributor to the decrease in the sensible heating source of dry entropy is the large increase in effective temperature. In the runs without scalar diffusion, the effective temperature of sensible heating is representative of the surface air temperature, which is within roughly 1 K of the imposed sea surface temperature. In the runs with scalar diffusion, the effective temperatures are much warmer than any temperature that is physically present in the model—in one of the runs, the effective temperature is almost 370 K.

These very high temperatures are consistent with the model formulation, and yet are entirely unphysical. In particular, large effective temperatures are produced by the repeated diffusion of heat from cold air to warm air. Mathematically, this may be shown by integrating the dry-entropy source by parts,

$$\int_V \left(-\frac{\nabla \cdot \mathbf{J}}{T}\right) = \int_S \frac{J^z}{T} - \int_V \frac{\mathbf{J} \cdot \nabla T}{T^2},$$

and then noting that the effective temperature of the first term on the right-hand side is the surface air temperature T_{surf}. If the second term on the right were zero, as it is in the simulations with no scalar diffusion, then the effective temperature of sensible heating would be T_{surf}. However, with diffusion of sensible heat in the bulk, the effective temperature becomes

$$T_J = \frac{T_{\mathrm{surf}}}{1 - T_{\mathrm{surf}} \dfrac{\int_V \mathbf{J} \cdot \nabla T/T^2}{\int_S J^z}},$$

which is greater than T_{surf} if $\mathbf{J}$ points up the temperature gradient.

In reality, of course, molecular processes diffuse and conduct heat down the temperature gradient. This apparent contradiction may be better understood by considering the physical process that models of subgrid diffusion are attempting to represent. Physically, the unresolved, subgrid turbulence in an unsaturated and statically stable atmosphere will push parcels down (up), causing them to warm (cool) adiabatically to a temperature greater (less) than their new surroundings. Turbulent mixing of the parcel with its new environment increases the temperature gradients to the point where molecular diffusion and conduction can take over and transport heat down the gradient. This process is a net source of dry entropy. However, at larger scales, the irreversible vertical exchange of two parcels in a stably stratified troposphere effectively transports heat from the higher and colder altitude (where a parcel of cold environmental air is exchanged for a parcel that is even colder due to adiabatic cooling) to the lower and warmer altitude (where a parcel of warm environmental air is exchanged for a parcel that is even warmer due to adiabatic warming). In the simulations, this process is modeled by an energy flux equal to minus the turbulent diffusivity times the gradient of the dry static energy, which leads to a transport of heat from cold to warm at the grid scales. This manifests itself in the model as a net sink of dry entropy. Therefore, unlike the modeled dry-entropy source from anemonal dissipation, which should be equal to the physical source so long as the grid scale is within the inertial subrange, the modeled dry-entropy source from diffusion of heat in the bulk is not even of the same sign as the true source at the scales of molecular diffusion and conduction.

Now, let us consider the effect of using a subgrid model of water-vapor diffusion. Because the specific humidity decreases with height, models of subgrid diffusion tend to move water vapor upward. This corresponds to a physical process that takes place in two steps. First, there is vertical eddy exchange: relatively dry air from above gets pushed down, and vice versa. Because there is no change in potential temperature, there is no change in dry entropy. Second, the parcel that has been displaced mixes with the environment, whereupon molecular diffusion transports water vapor from high partial pressures to low partial pressures. In the entropy budget, the entropy increase from diffusion down a partial-pressure gradient is captured explicitly. In the dry-entropy budget, there is no explicit source corresponding to this process. Instead, the effect of water-vapor diffusion is felt through a decrease in the condensation temperature; for the same amount of condensation heating, this amounts to an additional source of dry entropy. To see this effect on condensation temperature, we can approximate Eq. (25) as

$$\int_V e \log\left(\frac{q_v}{q_{v,\mathrm{surf}}}\right) \simeq -\int_V \mathbf{d}_v \cdot \boldsymbol{\nabla} \log(q_v), \qquad (27)$$

where $q_{v,\mathrm{surf}}$ is the effective mass fraction of surface air into which water vapor is diffused from the surface; that is,

$$\log(q_{v,\mathrm{surf}}) \equiv \frac{\int_S d_v^z \log(q_v)}{\int_S d_v^z}.$$

If there is no diffusion of water vapor in the bulk, then the right-hand side of Eq. (27) is zero and the effective mass-fraction of condensation must be equal to $q_{v,\mathrm{surf}}$. In other words, the only way for water vapor to enter and exit a Lagrangian parcel is by contact with the surface or by phase change, so the average q_v for one of these processes must equal the average q_v for the other. By introducing a nonzero eddy diffusivity, water vapor will diffuse down the gradient of q_v, making the right-hand side of Eq. (27) positive. Because the mean volume average of e is negative, the effective q_v on the left-hand side must be less than $q_{v,\mathrm{surf}}$. On average, q_v decreases with height both in the environment and within convective plumes, so a smaller effective water-vapor mass fraction of condensation implies that condensation is taking place at a lower effective temperature. Indeed, this prediction is born out by all four pairs of simulations presented in Table 3.

To simplify the physical interpretation of the dry-entropy budget, it is useful to combine several of the terms. Ignoring terms that are second order in $T/T_0 - 1$, which are on the order of a few tenths of one milliwatt per square meter per Kelvin or smaller, it is possible to write

$$-\frac{L_e(T)e}{T} + (c_{pv} - c_{vl})e \log\left(\frac{T}{T_0}\right) \simeq -\frac{L_e(T_0)e}{T}$$
$$-\frac{L_m(T)m}{T} + (c_{vl} - c_{vs})m \log\left(\frac{T}{T_0}\right) \simeq -\frac{L_m(T_0)m}{T}.$$

In other words, the source produced by the sensible heating by water as it moves through the atmosphere's temperature gradients is, to a good approximation, canceled by the temperature variations of L_e and L_m (Pauluis 2000).

The $-L_m(T_0)m/T$ term may be written as a sum of contributions from melting and fusion by taking advantage of the fact that the local melting rate is max(0, m) = (m + $|m|$)/2, and similarly for fusion. This allows us to separate the effects of melting and fusion as

$$-\int_V \frac{L_m(T_0)m}{T} = -\int_V \frac{L_m(T_0)(m + |m|)}{2T} - \int_V \frac{L_m(T_0)(m - |m|)}{2T}.$$

By defining effective temperatures of melting (T_m) and fusion (T_f), using the fact that $\int_V m = 0$, and neglecting terms that are second order in $T_f/T_m - 1$, this becomes

$$-\int_V \frac{L_m(T_0)m}{T} \simeq \frac{L_m(T_0)}{T_m^2} \Delta T \int_V \frac{|m|}{2}, \qquad (28)$$

where $\Delta T = T_m - T_f$ is the temperature separation between the melting and fusion. Because most of the melting takes place within close proximity of the melting line, T_m may be approximated by 273 K. Note also that $\int_V |m|/2$ is equal to the amount of water (in kilograms per second) that participates in the fuse/melt cycle. Equation (28) takes the same form as the conduction of sensible heat in the bulk: heat is transported at a rate $L_m(T_0)\int_V |m|/2$ across a temperature difference of ΔT. In effect, the fuse/melt cycle conducts heat from the warm melting line to the colder upper troposphere. We may characterize this as the atmosphere's "conductor."

The $\log(p)$ term may be rewritten (Pauluis 2000; Pauluis and Held 2002b) as

$$
\begin{aligned}
-\int_V R_v(e - \nabla\cdot\mathbf{d})\log(p/p_0) &= -\int_V R_v\partial_\alpha(q_v\rho u^\alpha)\log(p/p_0) \\
&\simeq \int_V \frac{R_v q_v \rho\mathbf{u}\cdot\nabla p}{p} \\
&\simeq \int_V \frac{R_v}{R_m}\frac{q_v\rho\mathbf{u}\cdot\mathbf{g}}{T}. \qquad (29)
\end{aligned}
$$

In the second line, the time derivative of pressure has been neglected, and in the third line the hydrostatic approximation has been used. In addition, it is possible to manipulate the precipitation–dissipation dry-entropy source as follows:

$$
\begin{aligned}
\int_V \frac{\mathbf{g}\cdot(\mathbf{d}_v+\mathbf{d}_l+\mathbf{d}_s)}{T} &\simeq \int_V \frac{\nabla p\cdot(\mathbf{d}_v+\mathbf{d}_l+\mathbf{d}_s)}{\rho T} \\
&= \int_V R_m\nabla\log(p/p_0)\cdot(\mathbf{d}_v+\mathbf{d}_l+\mathbf{d}_s) \\
&\simeq -\int_V R_m\log(p/p_0)\nabla\cdot(\mathbf{d}_v+\mathbf{d}_l+\mathbf{d}_s) \\
&\simeq \int_V R_m\log(p/p_0)\partial_\alpha[(q_v+q_l+q_s)\rho u^\alpha] \\
&\simeq -\int_V \frac{(q_v+q_l+q_s)\rho\mathbf{u}\cdot\nabla p}{p} \\
&\simeq -\int_V \frac{(q_v+q_l+q_s)\rho\mathbf{u}\cdot\mathbf{g}}{T}. \qquad (30)
\end{aligned}
$$

In the first and sixth lines, we have used the hydrostatic approximation. In the third line, we have used the assumption that deviations of p from p_0 are negligible at the surface and that gradients of R_m are negligible. In the fifth line, we have assumed that the time derivative of pressure may be neglected. As discussed in section 4 and by Pauluis et al. (2000), the space–time integrals of $d_v^z + d_l^z + d_s^z$ and $(q_v + q_l + q_s)\rho u^z$ must be equal. Therefore, the equality shown above guarantees that the effective temperatures for these two processes are the same. This is not an obvious fact because the temperatures of humid updrafts may, in general, differ by several degrees from the temperatures of precipitation-laden downdrafts. Adding the terms from Eqs. (29) and (30) gives

$$
\int_V \frac{\mathbf{g}\cdot(\mathbf{d}_v+\mathbf{d}_l+\mathbf{d}_s)}{T} - \int_V R_v e\log(p/p_0) \simeq -\int_V \frac{1}{T}[(1 - R_v/R_m)q_v + q_l + q_s]\rho\mathbf{u}\cdot\mathbf{g}. \qquad (31)
$$

Because the pressure of a moist parcel is $p = (1 + \epsilon q_v - q_l - q_s)R_a\rho T$, the fractional difference in density between a moist parcel and a dry parcel at the same pressure and temperature is

$$
\frac{\rho|_{p,T,\mathrm{moist}} - \rho|_{p,T,\mathrm{dry}}}{\rho} \simeq (1 - R_v/R_a)q_v + q_l + q_s,
$$

where terms that are second-order in the mass fractions have been neglected. Therefore, the right-hand side of (31) is the work performed by the atmosphere in lifting water, divided by the temperature. In other words, this is the amount of the dry-entropy sink generated by radiation, condensation, and sensible heating that is consumed by the atmosphere's need to act like an elevator, lifting water through the gravitational field. As a shorthand, we will refer to this as the atmosphere's "elevator."

The bulk-heating budget may now be approximated as

$$
\int_V \Bigg[\underbrace{\frac{Q}{T} - \frac{L_e(T_0)e}{T} - \frac{\nabla\cdot\mathbf{J}}{T}}_{\text{Rad/Cond/Sens Heating}} + \underbrace{\frac{\mathcal{D}}{T}}_{\text{Brakes}} + \underbrace{\frac{(q_l + q_s - \epsilon q_v)\rho w g}{T}}_{\text{Elevator}} + \underbrace{\frac{L_m(T_0)\Delta T}{T_m^2}\frac{|m|}{2}}_{\text{Conductor}}\Bigg] \simeq 0.
$$

Here, we use the term "brakes" as a shorthand for anemonal dissipation in reference to the role of friction in slowing down the atmosphere's motion. Note that this budget differs from Eq. (4) by the temperature independence of the latent heat of evaporation and the presence of the elevator and conductor terms. Calculating these terms for the numerical simulations, we get the dry-entropy budget in Table 4. As energy is moved through the atmosphere from the earth's surface to outer space, a net sink of dry entropy is produced by virtue of the fact that the temperatures at which the energy is added as heat (by sensible heating and condensation) are higher than the temperature at which the energy is removed (by radiation). This sink must be canceled by the sum of the next three terms: the frictional force acting to slow down the winds (the brakes), the work performed in lifting water (the elevator), and the transport of heat to the cold upper atmosphere by fusion and melting (the conductor). The analogy to an engine is straightforward. The radiation, condensation,

TABLE 4. The simplified dry-entropy budget (mW m^{-2} K^{-1}).

	5, 300 w/o diff	5, 303 w/o diff	10, 300 w/o diff	10, 303 w/o diff	5, 300 w/ diff	5, 303 w/ diff	10, 300 w/ diff	10, 303 w/ diff
Radiation	−402.6	−473.1	−432.2	−503.7	−397.9	−468.4	−426.4	−497.6
Condensation	367.9	431.1	411.3	477.3	366.1	429.8	408.0	473.6
Sensible	25.5	31.8	11.2	16.1	22.9	29.0	9.0	13.7
Brakes	5.3	6.0	5.8	6.5	4.6	5.2	5.0	5.6
Elevator	3.0	3.2	3.2	3.4	3.1	3.4	3.4	3.6
Conductor	1.7	1.7	1.7	1.7	1.8	1.8	1.8	1.8

and sensible heating act like the fuel, which generates a net sink of dry entropy. Part of that sink is wasted by the conductor, just like a combustion engine that experiences conductive losses. The remainder of the sink is used to maintain the atmosphere's motion against the frictional force of eddies (the brakes) and the weight of water (the elevator).

This picture leads to a set of four rather intuitive reasons for the low anemonal dissipation in moist convection as compared to dry convection. First, in dry convection, the supply of energy from the surface takes the form of sensible heat, which is added to the atmosphere at the surface air temperature. In moist convection, a large amount of the energy supplied from the surface takes the form of latent heat, which must rise up through the atmosphere to a higher altitude before it can condense. This decreases the temperature separation between the addition and removal of heat, which reduces the overall dry-entropy sink.

Next, imagine a dry atmosphere that has the same vertical profiles of heating and cooling as in a moist atmosphere. Although the moist atmosphere now has the same effective heights of heating and cooling as this fictional dry atmosphere, the temperature separations would still be much smaller because the lapse rate is nearly a moist adiabat instead of a dry adiabat. This difference in lapse rates is the second reason for the smaller dry-entropy sink produced by the energy flow through a moist atmosphere.

The third and fourth reasons for a smaller anemonal dissipation are evident in Table 4. The dry-entropy sink produced by sensible heating, condensation, and radiation must be shared among anemonal dissipation and two other terms that are not present in a dry atmosphere: the lifting of water by the atmosphere's water elevator and the transport of heat by the atmosphere's conductor. Therefore, the four key reasons for the low anemonal dissipation in moist convection are the small Bowen ratio, the moist lapse rate, the elevator, and the conductor.

It is notable that the low precipitation efficiency of moist precipitation does not make this list. In runs with previous versions of DAM that used the same specific heat capacity for all phases of water and employed a constant turbulent viscosity instead of a Smagorinsky scheme, the effective temperature of evaporation was consistently warmer than the effective temperature of condensation by about 2 K. Combined with a low precipitation efficiency, this temperature separation made the effective temperature of net condensation many degrees colder than either the temperature of condensation or evaporation. This effect played a pivotal role in bringing the effective temperature of net condensation closer to the radiation temperature. However, in the simulations presented here using realistic heat capacities and a Smagorinsky scheme, the effective temperatures of condensation and evaporation are nearly identical, so there is no such effect. The mean temperature difference between condensation and evaporation among the eight simulations is −0.01 K, with a standard deviation of 0.14 K. None of the eight runs had a temperature separation greater than 0.20 K. Because this demonstrates that the low precipitation efficiency cannot be trusted to reliably lower the condensation temperature—indeed, the low precipitation efficiency raised the condensation temperature in some runs—this effect cannot be added to the list of explanations for the small anemonal dissipation of moist convection.

An explanation for the small anemonal dissipation was given by Pauluis and Held (2002a,b) from the perspective of the entropy budget, which involved the diffusion of water vapor and nonequilibrium phase changes. Although they showed how these entropy sources could be rewritten approximately in terms of the condensation temperature, the intuition remained rooted in a water-vapor-centric perspective: for example, the work performed by water vapor and the transport of latent heat. On the other hand, the budget of dry entropy automatically gives a formulation in terms of the condensation temperature that can be derived exactly. In a sense, this cuts out the middleman of water vapor and presents a budget that includes only sensible heating and work, much like a textbook heat engine. The intuition gained from this perspective is not at odds with the intuition presented by Pauluis and Held (2002a,b), but it is arguably simpler to understand and more amenable to use as a predictive tool.

The effect of the Bowen ratio was discussed implicitly by Pauluis and Held (2002a) through their comparison of dry and moist atmospheres. The explicit inclusion of the small Bowen ratio as one of the four causes of small anemonal dissipation is warranted for a variety of reasons: observations that the Bowen ratio is small over the tropical ocean in both empirical studies and numerical simulations, the fact that a limit on the Bowen ratio may be derived theoretically, and the expectation that the Bowen ratio will change in a predictable way with global warming. The first of these three reasons is supported by the energy budgets presented in Table 1, where the Bowen ratio was always less than 0.1. It is also possible to put a theoretical upper limit on the Bowen ratio using a bulk surface-flux scheme. Over the ocean, the largest Bowen ratio will occur when the surface air is saturated. In this case, the sensible flux will be proportional to $c_{pm}\Delta T$ and the latent heat flux will be proportional to $L_e(T)[q^{*,l}(T) - q^{*,l}(T - \Delta T)]$. Therefore, the maximum Bowen ratio is given by

$$B_{\max} = \frac{c_{pm}}{L_e(T)}\left(\frac{\partial q_v^{*,l}}{\partial T}\right)^{-1}.$$

At sea level pressure, a sea surface temperature of 300 K gives a maximum Bowen ratio of 0.32. In reality and in the simulations, the surface air is undersaturated, so the Bowen ratio is significantly smaller than this. However, it is reasonable to assume that the Bowen ratio over the tropical ocean will change with global warming in the same direction as this upper limit, especially if the boundary layer relative humidity remains unchanged. Because this upper limit decreases with increasing temperature (0.25 for 305 K and 0.20 for 310 K), the Bowen ratio over the tropical ocean should decrease with increasing temperatures. This expectation is confirmed by studies of general circulation models (Held and Soden 2006). Therefore, the reduced Bowen ratio in a warmer world should predispose the anemonal dissipation toward smaller values.

The second cause of the smallness of dissipation is the relatively small lapse rate in a moist atmosphere. This point has not been emphasized in previous work, but it is certainly important enough to be included on the short list. Compared to an atmosphere with a 10 K km^{-1} lapse rate, an atmosphere with the same sensible heatings but a lapse rate of 6 K km^{-1} will experience a 40% reduction in the magnitude of the dry-entropy sink available to the brakes, elevator, and conductor. Expectations that the lapse rate will decrease with increased temperatures mean that this effect will incline the anemonal dissipation toward smaller values with global warming. Although both a smaller Bowen ratio and a smaller lapse rate would, all else equal, lead to a smaller value of dissipation, other effects, such as changes in the effective radiation temperature with increased CO_2, may actually lead to increases in dissipation.

The two terms that combine to make the elevator term, $-R_v e \log(p/p_0)$ and $\mathbf{g} \cdot (\mathbf{d}_l + \mathbf{d}_s)/T$, were discussed by Pauluis and Held (2002a), who described the first term as the work performed by water vapor. In their simulation, they found that the sum of these terms was a net source of entropy, but there was not enough data from the one simulation to say anything definite about the sign of the sum for general RCE. In the eight simulations shown here, the sum is consistently positive, leading to some confidence that the sum is positive for moist RCE under current tropical oceanic conditions. Because the value of the elevator source is between 50% and 70% as large as the source from anemonal dissipation, it plays an important role in reducing the amount of dissipation that can be performed by a moist atmosphere. As for the conductor, its effect on the entropy and dry-entropy budgets has not previously been addressed. As seen in Table 4, it is large enough to be of importance when quantitative estimates of dissipation are desired.

As we have seen, the dry-entropy budget may be of some use in predicting the vigor of convection under global-warming scenarios. The dry-entropy budget may also predict changes in anemonal dissipation in numerical models given various changes in the parameterizations of radiation and microphysics. For example, a change in the radiation scheme that favors radiative cooling from higher (and, therefore, colder) altitudes will, all else being equal, increase the anemonal dissipation. Similarly, a decrease in the shortwave absorption by high clouds should also be expected to increase the dissipation by reducing the radiative heating at low temperatures.

As for the microphysics, a particularly simple set of parameters to consider would be the terminal velocities of condensates. As the terminal velocities are increased, the residence times of liquid and solid water will decrease, which will lead to a decreased source of dry entropy from the lifting of water. The amount of water participating in the fuse/melt cycle might also decrease because more liquid condensate will have rained out before reaching the fusion height. If the dry-entropy sink from sensible heating, condensation, and radiation is unchanged, then the source from anemonal dissipation must increase. For sufficiently large terminal velocities of the condensates, the buoyancy from water vapor may perform more work than is required to lift the condensate during its short lifetime in the updrafts. In this case, the lifting of water will become a net sink of dry entropy. As terminal velocities approach infinity, the elevator term asymptotes to a sink equal to the integral of the buoyant work by water vapor divided by the temperature. This suggests that the anemonal dissipation should asymptote to a larger value as terminal velocities are increased, which is in agreement with numerical simulations by Parodi and Emanuel (2006).

It is also possible to gain predictive power from the dry-entropy budget by combining it with empirical observations. For example, the numerical simulations suggest that the anemonal dissipation may be modeled as a fixed fraction of the radiative cooling, regardless of surface temperature or wind speed. The fraction $-\int_V \mathcal{D}/\int_V Q$ took a mean value of 0.0131 for the runs without scalar diffusion and a mean value of 0.0115 with scalar diffusion. Although the values of the anemonal dissipation varied by as much as 24% [(max − min)/min] within each set of four runs, the ratios of dissipation to radiative cooling varied by no more than 4.5%. This finding appears to concur with the study by Robe and Emanuel (1996), who found in their cloud-resolving simulations that the convective mass flux increased almost linearly with the total radiative cooling, whereas the vertical velocity of the updrafts remained unchanged. If the anemonal dissipation is a function only of the vertical velocity profile of updrafts and the number of updrafts, then their results would imply a fixed ratio of anemonal dissipation to radiative cooling. Because the effective temperature of dissipation does not vary significantly, the ratio $-\int_V (\mathcal{D}/T)/\int_V Q$ is also approximately invariant. In addition, the source of dry entropy from the elevator also scales very well with the total radiative cooling. On the other hand, the conductor does not scale with the radiative cooling, but it is the smallest of the three entropy sources. Therefore, to a good approximation, the net sink of dry entropy from sensible heating, condensation, and radiation is an invariant fraction of the total radiative cooling.

For the simulations without scalar diffusion, this invariance may be written as

$$\frac{1}{\int_V Q}\left(\frac{\int_V Q}{T_Q} - \frac{\int_V L_e e}{T_c} - \frac{\boldsymbol{\nabla}\cdot\mathbf{J}}{T_{\text{surf}}}\right) = \alpha,$$

where α is an empirically determined constant, T_Q is the effective temperature of radiative cooling, T_c is the effective temperature of condensation, and the lack of heat diffusion justifies the substitution of T_{surf} for the effective temperature of sensible heating. This may be rewritten in terms of the Bowen ratio B as

$$\frac{1}{(1+B)T_c} + \frac{B}{(1+B)T_{\text{surf}}} - \frac{1}{T_Q} + \alpha = 0.$$

This approximation provides very good estimates of T_c given the other values but does less well when used to estimate the Bowen ratio. Among the eight simulations, the Bowen ratio ranges from 2.7% to 7.6%. As discussed earlier, the smallness of this ratio explains a large amount of the difference in anemonal dissipation between dry and moist convection. It is less clear, however, what role the Bowen ratio plays in regulating the anemonal dissipation for moist convection under different forcings. If it is true that the anemonal dissipation is a fixed fraction of radiative cooling, then the Bowen ratio may play more of a supporting role, adjusting as necessary to bring the radiation/condensation/sensible dry-entropy sink in line with the brakes/elevator/conductor dry-entropy source.

7. Conclusions

By studying the dry-entropy budget of a moist atmosphere, including a solid phase of water, a simple analogy to a heat engine has emerged. The heating from radiation, condensation, and sensible fluxes provide a sink that must be matched by the sources corresponding to the work performed against wind-generated frictional dissipation (the "brakes"), the work performed in lifting water (the "elevator"), and the heat transport from the fuse/melt cycle in the upper troposphere (the "conductor"). The small Bowen ratio and small lapse rate of moist convection explain why the available sink from radiation, condensation, and sensible heat fluxes must be small. The fact that this small sink must be shared with the lifting of water and the fuse/melt cycle further reduces the size of the wind-generated dissipation.

Indications from numerical simulations suggest that terms in the dry-entropy budget may change in understandable and predictable ways given changes in forcing or parameterization. For example, it was shown that an increase in the eddy diffusivity of water vapor will lower the effective temperature of condensation. In addition, empirical observations may be coupled with the dry-entropy budget to find powerful constraints on effective temperatures and the Bowen ratio. The observation that the wind-generated dissipation increases linearly with the total radiative cooling is an example of such an input. Furthermore, the dry-entropy budget is only one out of many possible integral equations that could potentially yield powerful insights into the steady-state behavior of moist convection.

Several questions and avenues of inquiry are raised by the results presented here. For example, while there are many different formulations of subgrid processes in the literature on fluid turbulence, they have not yet been systematically subjected to scrutiny from the perspective of the entropy or dry-entropy budgets. As shown here, the first-order down-gradient Smagorinsky formulation for the subgrid transport of dry static energy clearly fails to comply with expectations that subgrid heat transport be a net source of entropy or dry entropy. It would merit further research to assess existing turbulence schemes for their ability to produce entropy sources of the correct sign and magnitude.

To date, cloud-resolving studies of the moist entropy budget have been restricted to disorganized convection. It remains to be seen what kind of impact mesoscale organization might have on the terms in the entropy budget. How different is the rate of dissipation in a tropical cyclone compared to disorganized convection, and what differences in the entropy budget allow for that difference? Can the entropy budget place stringent bounds on the intensity of storms? Questions like these have not yet been addressed, but given the importance of wind-generated dissipation to those living in the atmosphere's boundary layer, these questions certainly deserve attention.

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APPENDIX A

Deriving the Dry-Entropy Budget (21)

The integrand in (20),

$$\partial_\alpha\{\rho u^\alpha[c_{pm}\log(T/T_0) - R_m\log(p/p_0)]\},$$

can be shown to equal the integrand in (21) by using the differential governing equations. Ignoring the logarithms for the moment, we can derive

$$\begin{aligned}\partial_\alpha(c_{pm}\rho u^\alpha) &= c_{pa}\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha] + c_{pv}\partial_\alpha(q_v\rho u^\alpha) + c_{vl}\partial_\alpha(q_l\rho u^\alpha) + c_{vs}\partial_\alpha(q_s\rho u^\alpha)\\ &= c_{pv}(e - \nabla\cdot\mathbf{d}_v) + c_{vl}(-e + m - \nabla\cdot\mathbf{d}_l) + c_{vs}(-m - \nabla\cdot\mathbf{d}_s),\end{aligned}$$

and

$$\partial_\alpha(R_m\rho u^\alpha) = R_a\partial_\alpha[(1 - q_v - q_l - q_s)\rho u^\alpha] + R_v\partial_\alpha(q_v\rho u^\alpha) = R_v(e - \nabla\cdot\mathbf{d}_v).$$

Therefore, the divergence of the first term is

$$\begin{aligned}\partial_\alpha[c_{pm}\rho u^\alpha\log(T/T_0)] &= [c_{pv}(e - \nabla\cdot\mathbf{d}_v) + c_{vl}(-e + m - \nabla\cdot\mathbf{d}_l) + c_{vs}(-m - \nabla\cdot\mathbf{d}_s)]\log(T/T_0) + c_{pm}\rho u^\alpha\frac{1}{T}\partial_\alpha T\\ &= [c_{pv}(e - \nabla\cdot\mathbf{d}_v) + c_{vl}(-e + m - \nabla\cdot\mathbf{d}_l) + c_{vs}(-m - \nabla\cdot\mathbf{d}_s)]\log(T/T_0) + \frac{c_{pm}}{T}\partial_\alpha(\rho T u^\alpha)\\ &\quad - c_{pm}\partial_\alpha(\rho u^\alpha)\\ &= [c_{pv}(e - \nabla\cdot\mathbf{d}_v) + c_{vl}(-e + m - \nabla\cdot\mathbf{d}_l) + c_{vs}(-m - \nabla\cdot\mathbf{d}_s)]\log(T/T_0)\\ &\quad + c_{pm}\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)\\ &\quad + \frac{c_{pm}}{c_{vm}T}[Q - p\nabla\cdot\mathbf{u} + \mathcal{D} - c_{vv}\mathbf{d}_v\cdot\nabla T - \nabla\cdot(R_v T\mathbf{d}_v) - c_{vl}\mathbf{d}_l\cdot\nabla T - c_{vs}\mathbf{d}_s\cdot\nabla T\\ &\quad + \mathbf{g}\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s) - \nabla\cdot\mathbf{J} - eE_{0v} - mE_{0s} - c_{vm}T\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)\\ &\quad + (c_{vl} - c_{vv})(T - T_{\mathrm{trip}})e + (c_{vs} - c_{vl})(T - T_{\mathrm{trip}})m],\end{aligned}$$

where the last line uses the expression for $\partial_\alpha(c_v T\rho u^\alpha)$, which is obtained by subtracting from (11) the sum of (17), E_{0v} times (7), and $-E_{0s}$ times (9). The divergence of the negative of the second term is

$$\begin{aligned}\partial_\alpha[R_m\rho u^\alpha \log(p/p_0)] &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(p/p_0) + R_m\rho u^\alpha\frac{1}{p}\partial_\alpha p\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(p/p_0) + \frac{u^\alpha}{T}\partial_\alpha(R_m\rho T)\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(p/p_0) + \rho u^\alpha\partial_\alpha R_m + \frac{u^\alpha R_m}{T}\partial_\alpha(\rho T)\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(p/p_0) + \partial_\alpha(R_m\rho u^\alpha) - R_m\partial_\alpha(\rho u^\alpha)\\ &\quad + \frac{R_m}{T}\partial_\alpha(T\rho u^\alpha) - R_m\rho\nabla\cdot\mathbf{u}\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(p/p_0) + R_v(e - \nabla\cdot\mathbf{d}_v) + R_m\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)\\ &\quad - \frac{p}{T}\nabla\cdot\mathbf{u} + \frac{R_m}{c_{vm}T}[Q - p\nabla\cdot\mathbf{u} + \mathcal{D} - c_{vv}\mathbf{d}_v\cdot\nabla T - \nabla\cdot(R_vT\mathbf{d}_v)\\ &\quad - c_{vl}\mathbf{d}_l\cdot\nabla T - c_{vs}\mathbf{d}_s\cdot\nabla T + \mathbf{g}\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s) - \nabla\cdot\mathbf{J} - eE_{0v}\\ &\quad - mE_{0s} - c_{vm}T\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s) + (c_{vl} - c_{vv})(T - T_{\mathrm{trip}})e\\ &\quad + (c_{vs} - c_{vl})(T - T_{\mathrm{trip}})m].\end{aligned}$$

Subtracting this expression from the first one and integrating over space and time yields Eq. (21).

APPENDIX B

Deriving the Integral Equation (25)

We will show here that the integral Eq. (25) corresponds to the specific quantity

$$R_m\log(1 + \epsilon q_v - q_l - q_s) - R_v q_v\log[(1 + \epsilon)q_v].$$

The space–time integral of ρu^α times the first term gives

$$\begin{aligned}\partial_\alpha[\rho u^\alpha R_m\log(1 + \epsilon q_v - q_l - q_s)] &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(1 + \epsilon q_v - q_l - q_s) + R_a\rho u^\alpha\partial_\alpha(1 + \epsilon q_v - q_l - q_s)\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(1 + \epsilon q_v - q_l - q_s) + R_a\epsilon\partial_\alpha(q_v\rho u^\alpha) - R_a\epsilon q_v\partial_\alpha(\rho u^\alpha)\\ &\quad - R_a\partial_\alpha(q_l\rho u^\alpha) + R_a q_l\partial_\alpha(\rho u^\alpha) - R_a\partial_\alpha(q_s\rho u^\alpha) + R_a q_s\partial_\alpha(\rho u^\alpha)\\ &= R_v(e - \nabla\cdot\mathbf{d}_v)\log(1 + \epsilon q_v - q_l - q_s) + R_a(\epsilon q_v - q_l - q_s)\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s)\\ &\quad + R_a\epsilon(e - \nabla\cdot\mathbf{d}_v) - R_a(-e + m - \nabla\cdot\mathbf{d}_l) - R_a(-m - \nabla\cdot\mathbf{d}_s).\end{aligned}$$

Similarly, the second term gives

$$\begin{aligned}-\partial_\alpha\{\rho u^\alpha R_v q_v\log[(1 + \epsilon)q_v]\} &= -R_v(e - \nabla\cdot\mathbf{d}_v)\log[(1 + \epsilon)q_v] - R_v\rho u^\alpha\partial_\alpha q_v\\ &= -R_v(e - \nabla\cdot\mathbf{d}_v)\log[(1 + \epsilon)q_v] - R_v\partial_\alpha(q_v\rho u^\alpha) + R_v q_v\partial_\alpha(\rho u^\alpha)\\ &= -R_v(e - \nabla\cdot\mathbf{d}_v)\log[(1 + \epsilon)q_v] - R_v(e - \nabla\cdot\mathbf{d}_v) - R_v q_v\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s).\end{aligned}$$

Adding the two terms, integrating, and recalling that the integral of a space–time divergence is zero yields

$$\int_V R_v(e - \nabla\cdot\mathbf{d}_v)\log\left[\frac{1 + \epsilon q_v - q_l - q_s}{(1 + \epsilon)q_v}\right] + \int_V R_a(1 - q_v - q_l - q_s)\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s) = 0. \tag{B1}$$

Note that the second term in (B1) may be written as

$$-\int_V R_a(1 - q_v - q_l - q_s)\partial_\alpha(\rho u^\alpha).$$

Using the fact that

$$\partial_\alpha(\rho u^\alpha) = \partial_\alpha[(q_v + q_l + q_s)\rho u^\alpha],$$

and assuming a steady state so that the time-averaged space–time integral of a four-divergence is zero, we can write, for any real number n,

$$\begin{aligned}\int_V (q_v + q_l + q_s)^n\partial_\alpha(\rho u^\alpha) &= \int_V (q_v + q_l + q_s)^n\partial_\alpha[(q_v + q_l + q_s)\rho u^\alpha] \\ &= -\int_V (q_v + q_l + q_s)\rho u^\alpha\partial_\alpha[(q_v + q_l + q_s)^n] \\ &= -n\int_V (q_v + q_l + q_s)^n\rho u^\alpha\partial_\alpha(q_v + q_l + q_s) \\ &= -n\int_V (q_v + q_l + q_s)^n\partial_\alpha[(q_v + q_l + q_s)\rho u^\alpha] \\ &\quad + n\int_V (q_v + q_l + q_s)^{n+1}\partial_\alpha(\rho u^\alpha) \\ &= -n\int_V (q_v + q_l + q_s)^n\partial_\alpha(\rho u^\alpha) + n\int_V (q_v + q_l + q_s)^{n+1}\partial_\alpha(\rho u^\alpha),\end{aligned}$$

which can be rearranged for any $n \neq -1$ to give

$$\int_V (q_v + q_l + q_s)^n\partial_\alpha(\rho u^\alpha) = \frac{n}{n+1}\int_V (q_v + q_l + q_s)^{n+1}\partial_\alpha(\rho u^\alpha).$$

By induction, we have, for any nonnegative integer m and any positive integer n,

$$\int_V (q_v + q_l + q_s)^m\partial_\alpha(\rho u^\alpha) = \frac{m}{n}\int_V (q_v + q_l + q_s)^n\partial_\alpha(\rho u^\alpha).$$

By taking the limit of large n and noting that $[(q_v + q_l + q_s)^n]/n$ goes to zero in this limit because $q_v + q_l + q_s \leq 1$, we get

$$\int_V (q_v + q_l + q_s)^m\partial_\alpha(\rho u^\alpha) = 0. \qquad \text{(B2)}$$

In a similar fashion, it is possible to demonstrate the identities

$$\int_V f(q_v + q_l + q_s)\nabla\cdot(\mathbf{d}_v + \mathbf{d}_l + \mathbf{d}_s) = 0$$

$$\int_V f\left(\frac{q_v}{1 - q_l - q_s}\right)(e - \nabla\cdot\mathbf{d}_v) = 0$$

$$\int_V f\left(\frac{q_l}{1 - q_v - q_s}\right)(-e + m - \nabla\cdot\mathbf{d}_l) = 0$$

$$\int_V f\left(\frac{q_s}{1 - q_v - q_l}\right)(-m - \nabla\cdot\mathbf{d}_s) = 0$$

for any sufficiently well-behaved function f. From Eq. (B2) with $m = 0$ and $m = 1$, it is clear that the second term in (B1) is zero. Therefore, we get the desired integral equation,

$$\int_V R_v(e - \nabla\cdot\mathbf{d}_v)\log\left[\frac{1 + \epsilon q_v - q_l - q_s}{(1 + \epsilon)q_v}\right] = 0.$$

Multiplying the numerator and denominator of the fraction by $R_a\rho T$ yields Eq. (25).

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