Das Atmosphärische Modell (DAM)

Model Integration

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DAM uses a split-explicit time stepping approach to acoustic modes and gravity waves. The terms in the governing equations that are responsible for sound waves and gravity waves are integrated with a small time step (the acoustic loop) while the other terms are integrated with a larger time step (the Runge-Kutta loop).

1 The Runge-Kutta loop

The large time step is integrated using the Runge-Kutta (RK) method. Four RK schemes are implemented in DAM with the following Butcher tables,

$$
\begin{aligned}
\text{butcher\_RK1} &= 1 \\
\text{butcher\_RK2} &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \\
\text{butcher\_RK3} &= \begin{pmatrix} 1/2 & 0 & 0 \\ -1 & 2 & 0 \\ 1/6 & 2/3 & 1/6 \end{pmatrix} \\
\text{butcher\_RK4} &= \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/6 & 1/3 & 1/3 & 1/6 \end{pmatrix} .
\end{aligned}
$$

The Runge-Kutta order (`rk_order`) is set in the `params_nml` namelist in the `nml` file.

The Runge-Kutta method is implemented in the `rk_update` subroutine in `rk_mod.f90`. The sequence of steps in `rk_update` is as follows:

- Save state variables u, v, w, θ_v, etc. (`u`, `v`, `w`, `t`, etc.) as "`_last`" variables (`u_last`, `v_last`, `w_last`, `t_last`, etc.).
- Begin the RK loop, looping over `rk_step` from 1 to `rk_order`.
  - Calculate the tendencies of ρu, ρv, ρw, $\rho\theta_v$, and ρ and store them in `SU`, `SV`, `SW`, `ST`, and `SRHO`.
  - Call `acoustic_loop`, which stores the changes in ρu, ρv, ρw, $\rho\theta_v$, and ρ during the time step dt in `UU`, `VV`, `WW`, `TT`, and `RHORHO`.
  - Copy those changes to the arrays `UVWTRK` and `RHORK`.
  - Calculate the changes in $q_v\rho$, $q_c\rho$, $q_r\rho$, $q_i\rho$, $q_s\rho$, and $q_g\rho$ during time step dt and store them in `QRK`.
  - Assign `rho` the value of `rho_last` and assign the other state variables `u`, `v`, `w`, `t`, etc. their respective densities before the RK loop, i.e., `rho_last` times `u_last`, `v_last`, `w_last`, `t_last`, etc.
  - Add to these state variables the changes recorded in `UVWTRK` and `RHORK` weighted by the `rk_order` row of the Butcher table. These are the updated densities.
  - Divide the state variables (other than `rho`) by `rho` to recover the updated specific (i.e., per mass) quantities.
  - Update the diagnostic variables like `p` and `tabs` (the absolute temperature).

2 The acoustic loop

The acoustic time step is performed with forward-backward time differencing in the manner described by Klemp et al. (2007). In particular, terms with horizontal derivatives are included in a forward sense in the horizontal momentum equations and are included in a backward sense in the equations for density and equivalent potential temperature. When the vertical grid spacing is smaller than the horizontal grid spacing, the terms involved in vertical sound propagation and gravity waves are treated implicitly.

During the acoustic loop, only the momentum densities (U, V, and W), the virtual potential temperature density (Θ), and the mass density (ρ) are updated. We will use the notation whereby the values of these variables at the beginning of the acoustic loop will be denoted by a superscript t and their deviation from that value at the current acoustic step will be denoted by a superscript number corresponding to the acoustic time step. For example,

$$U \text{ at acoustic step n } = U^t + U^n .$$

Schematically, the acoustic step can be written as

$$\begin{aligned}
U^2 &= U^1 + (\text{pressure gradient})^1 \\
V^2 &= V^1 + (\text{pressure gradient})^1 \\
W^2 &= W^1 + \frac{1}{2}(\text{pressure gradient plus gravity})^2 + \frac{1}{2}(\text{pressure gradient plus gravity})^1 \\
\Theta^2 &= \Theta^1 + (\text{horizontal flux})^2 + \frac{1}{2}(\text{vertical flux})^2 + \frac{1}{2}(\text{vertical flux})^1 \\
\rho^2 &= \rho^1 + (\text{horizontal flux})^2 + \frac{1}{2}(\text{vertical flux})^2 + \frac{1}{2}(\text{vertical flux})^1 .
\end{aligned}$$

For simplicity, the tendencies from terms that do not change during the acoustic step (S_U, S_V, S_W, S_Θ, and S_ρ) have been omitted. Note that the pressure gradient terms on the right-hand side of the U and V equations contain horizontal derivatives and are evaluated at step 1 (forward differencing). Meanwhile, the terms in the Θ and ρ equations that have horizontal derivatives (the horizontal flux terms) are evaluated at step 2 (backward differencing). The remaining terms (the vertical flux of Θ, the vertical pressure gradient, and gravitational acceleration) that play a role in vertical sound waves and gravity waves are treated implicitly, i.e., they are evaluated at a time linearly interpolated between steps 1 and 2.

Since the diagnostic variable p depends nonlinearly on the virtual potential temperature density, it must be Taylor expanded to first order in Θ^n,

$$\begin{aligned}
p^t + p^1 &= p_0 \left(\frac{R(\Theta^t + \Theta^1)}{p_0} \right)^{\gamma^t} \\
&= p_0 \left(\frac{R\Theta^t}{p_0} + \frac{R\Theta^1}{p_0} \right)^{\gamma^t} \\
&\approx p_0 \left(\frac{R\Theta^t}{p_0} \right)^{\gamma^t} + p_0 \gamma^t \left(\frac{R\Theta^t}{p_0} \right)^{\gamma^t - 1} \frac{R\Theta^1}{p_0} \\
&= p^t + \frac{p^t \gamma^t}{\Theta^t} \Theta^1 .
\end{aligned}$$

Similarly, for p^2, we have

$$p^2 = \frac{\gamma^t p^t}{\Theta^t} \Theta^2 .$$

Therefore, the implicit, vertical pressure gradient in the W equation may be written as

$$-\frac{1+\beta_s}{2} \frac{\partial}{\partial z} p^2 - \frac{1-\beta_s}{2} \frac{\partial}{\partial z} p^1 = -\frac{\partial}{\partial z} p^t - \frac{1+\beta_s}{2} \frac{\partial}{\partial z} \left(\frac{\gamma^t p^t}{\Theta^t} \Theta^2 \right) - \frac{1-\beta_s}{2} \frac{\partial}{\partial z} \left(\frac{\gamma^t p^t}{\Theta^t} \Theta^1 \right) .$$

The $-\partial p^t / \partial z$ term, which depends only on the pressure at the last large time step, is absorbed into the large-time-step tendency S_W.

To improve the numerical stability of this scheme, acoustic modes are filtered by forward centering of the vertically implicit terms (Durran and Klemp, 1983) and the horizontal pressure gradient in the U and V equations (Skamarock and Klemp, 1992; Klemp et al., 2007). Forward centering the implicit terms is accomplished by introducing a positive constant β_s such that the linearly interpolated terms are evaluated closer to step 2 than step 1,

$$\frac{1}{2}()^2 + \frac{1}{2}()^1 \;\longrightarrow\; \frac{1+\beta_s}{2}()^2 + \frac{1-\beta_s}{2}()^1 \,.$$

Since we can not use Θ^2 in the U and V equations without greatly complicating the implicit equations, the horizontal pressure gradient is forward centered by using the value from step 0,

$$\frac{\partial}{\partial x}\left(\frac{p^t\gamma^t}{\Theta^t}\Theta^1\right) \;\longrightarrow\; \frac{\partial}{\partial x}\left(\frac{p^t\gamma^t}{\Theta^t}\left(\Theta^1 + \beta_d\left(\Theta^1 - \Theta^0\right)\right)\right) .$$

The values of β_d and β_s can be set by the user in the `params_nml` namelist. Values of $\beta_d = \beta_s = 0.1$ are recommended Klemp et al. (2007) and are set as the default.

Denoting the acoustic time step size by dtn, the full acoustic-step equations are

$$\begin{aligned}
U^2 &= U^1 - \mathrm{dtn}\,\partial_x\left(\frac{p^t\gamma^t}{\Theta^t}\left(\Theta^1 + \beta_d(\Theta^1 - \Theta^0)\right)\right) + \mathrm{dtn}\,S_U \\
V^2 &= V^1 - \mathrm{dtn}\,\partial_y\left(\frac{p^t\gamma^t}{\Theta^t}\left(\Theta^1 + \beta_d(\Theta^1 - \Theta^0)\right)\right) + \mathrm{dtn}\,S_V \\
W^2 &= W^1 + \mathrm{dtn}\Bigg(- \frac{1+\beta_s}{2}\partial_z\left[\frac{\gamma^t p^t}{\Theta^t}\Theta^2\right] - \frac{1-\beta_s}{2}\partial_z\left[\frac{\gamma^t p^t}{\Theta^t}\Theta^1\right] - \frac{1+\beta_s}{2} g\rho^2 - \frac{1-\beta_s}{2} g\rho^1\Bigg) + \mathrm{dtn}\,S_W \\
\Theta^2 &= \Theta^1 + \mathrm{dtn}\left(-\partial_x\left(U^2\Theta^t\right) - \partial_y\left(V^2\Theta^t\right) - \frac{1+\beta_s}{2}\partial_z\left(W^2\Theta^t\right) - \frac{1-\beta_s}{2}\partial_z\left(W^1\Theta^t\right)\right) + \mathrm{dtn}\,S_\Theta \\
\rho^2 &= \rho^1 + \mathrm{dtn}\left(-\partial_x U^2 - \partial_y V^2 - \frac{1+\beta_s}{2}\partial_z W^2 - \frac{1-\beta_s}{2}\partial_z W^1\right) + \mathrm{dtn}\,S_\rho \,.
\end{aligned}$$

A single equation for W^2 may be obtained by substituting the definitions of Θ^2 and ρ^2 from the last two equations into the third equation. This gives

$$\begin{aligned}
W^2 = \; & W^1 + \mathrm{dtn}\Bigg(- \partial_z\left(\frac{\gamma^t p^t}{\Theta^t}\Theta^1\right) - g\rho^1 + S_W \\
& -\mathrm{dtn}\frac{1+\beta_s}{2}\partial_z\left\{\frac{\gamma^t p^t}{\Theta^t}\left[-\partial_x\left(U^2\Theta^t\right) - \partial_y\left(V^2\Theta^t\right) - \frac{1+\beta_s}{2}\partial_z\left(W^2\Theta^t\right) - \frac{1-\beta_s}{2}\partial_z\left(W^1\Theta^t\right) + S_\Theta\right]\right\} \\
& -\mathrm{dtn}\frac{1+\beta_s}{2} g\left[-\partial_x U^2 - \partial_y V^2 - \frac{1+\beta_s}{2}\partial_z W^2 - \frac{1-\beta_s}{2}\partial_z W^1 + S_\rho\right]\Bigg) .
\end{aligned}$$

Since U^2 and V^2 are known from their forward-differencing equations, only W^2 is unknown here. However, this is an implicit differential equation for W^2 and so must be solved by inverting matrices after this has been discretized. A second-order discretization is used here because it entails the inverting of a tridiagonal matrix, which is a relatively inexpensive operation; higher-order discretizations would require more expensive matrix operations. Using a subscript to denote the vertical level, the discretization can be written as

$$
\begin{aligned}
W_k^2 - \left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 \frac{1}{\mathrm{z}_k - \mathrm{z}_{km}} \Bigg\{ & \frac{1}{\mathrm{zi}_{kp} - \mathrm{zi}_k} \frac{\gamma_k^t p_k^t}{\Theta_k^t} \left(W_{kp}^2 \theta_{k/kp}^t - W_k^2 \theta_{km/k}^t \right) - \\
& \frac{1}{\mathrm{zi}_k - \mathrm{zi}_{km}} \frac{\gamma_{km}^t p_{km}^t}{\Theta_{km}^t} \left(W_k^2 \theta_{km/k}^t - W_{km}^2 \theta_{kmm/km}^t \right) \Bigg\} \\
& - \left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 g \frac{1}{2} \left(\frac{W_{kp}^2 - W_k^2}{\mathrm{zi}_{kp} - \mathrm{zi}_k} + \frac{W_k^2 - W_{km}^2}{\mathrm{zi}_k - \mathrm{zi}_{km}} \right) \\
& = \Bigg\{ W^1 + \mathrm{dtn}\Bigg(- \partial_z \left(\frac{\gamma^t p^t}{\Theta^t} \Theta^1 \right) - g\rho^1 + S_W \\
- \mathrm{dtn}\frac{1+\beta_s}{2} \partial_z & \left\{ \frac{\gamma^t p^t}{\Theta^t} \left[-\partial_x \left(U^2 \Theta^t \right) - \partial_y \left(V^2 \Theta^t \right) - \frac{1-\beta_s}{2} \partial_z \left(W^1 \Theta^t \right) + S_\Theta \right] \right\} \\
& - \mathrm{dtn}\frac{1+\beta_s}{2} g \left[-\partial_x U^2 - \partial_y V^2 - \frac{1-\beta_s}{2} \partial_z W^1 + S_\rho \right] \Bigg) \Bigg\}_k
\end{aligned}
$$

where z_k is the level-k scalar height, zi_k is the level-k interface height, $km = k-1$, $kmm = k-2$, $kp = k+1$, and

$$
\theta_{km/k}^t = \begin{cases} \text{if upwind} & \begin{cases} \theta_{km}^t & \text{if } W_k^1 > 0 \\ \theta_k^t & \text{if } W_k^1 \le 0 \end{cases} \\ \text{if centered} & \frac{1}{2}(\theta_{km}^t + \theta_k^t) \end{cases}
$$

Note that, since W^2 is the unknown, upwinding is done with W^1.

This set of coupled equations can be rewritten as

$$
c_k W_{kp}^2 + b_k W_k^2 + a_k W_{km}^2 = d_k
$$

where

$$
\begin{aligned}
c_k &= -\left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 (\mathrm{zi}_{kp} - \mathrm{zi}_k)^{-1} \left((\mathrm{z}_k - \mathrm{z}_{km})^{-1} \frac{\gamma_{m,k}^t p_k^t}{\Theta_k^t} \theta_{k/kp}^t + \frac{1}{2} g \right) \\
b_k &= 1 + \left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 (\mathrm{z}_k - \mathrm{z}_{km})^{-1} \theta_{km/k}^t \left[(\mathrm{zi}_{kp} - \mathrm{zi}_k)^{-1} \frac{\gamma_k^t p_k^t}{\Theta_k^t} + (\mathrm{zi}_k - \mathrm{zi}_{km})^{-1} \frac{\gamma_{km}^t p_{km}^t}{\Theta_{km}^t} \right] \\
&\quad + \left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 \frac{1}{2} g \left[(\mathrm{zi}_{kp} - \mathrm{zi}_k)^{-1} - (\mathrm{zi}_k - \mathrm{zi}_{km})^{-1} \right] \\
a_k &= -\left(\mathrm{dtn}\frac{1+\beta_s}{2}\right)^2 (\mathrm{zi}_k - \mathrm{zi}_{km})^{-1} \left((\mathrm{z}_k - \mathrm{z}_{km})^{-1} \frac{\gamma_{km}^t p_{km}^t}{\Theta_{km}^t} \theta_{kmm/km}^t - \frac{1}{2} g \right) \\
d_k &= \Bigg\{ W^1 + \mathrm{dtn}\Bigg(- \partial_z \left(\frac{\gamma^t p^t}{\Theta^t} \Theta^1 \right) - g\rho^1 + S_W \\
&\quad - \mathrm{dtn}\frac{1+\beta_s}{2} \partial_z \left\{ \frac{\gamma^t p^t}{\Theta^t} \left[-\partial_x \left(U^2 \Theta^t \right) - \partial_y \left(V^2 \Theta^t \right) - \frac{1-\beta_s}{2} \partial_z \left(W^1 \Theta^t \right) + S_\Theta \right] \right\} \\
&\quad - \mathrm{dtn}\frac{1+\beta_s}{2} g \left[-\partial_x U^2 - \partial_y V^2 - \frac{1-\beta_s}{2} \partial_z W^1 + S_\rho \right] \Bigg) \Bigg\}_k .
\end{aligned}
$$

The boundary conditions on W at the upper and lower boundaries are that W is zero there, which implies

$$
b_1 = b_{nz} = 1\,, \quad c_1 = a_{nz} = d_1 = d_{nz} = 0\,.
$$

In matrix notation, the set of coupled equations can be written in matrix notation as

$$
\begin{pmatrix} 1 & 0 & 0 & & & & \\ a_2 & b_2 & c_2 & & & & \\ & & & \ddots & & & \\ & & & & a_{nz-1} & b_{nz-1} & c_{nz-1} \\ & & & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W_1^2 \\ \\ \vdots \\ \\ W_{nz}^2 \end{pmatrix} = \begin{pmatrix} d_1 \\ \\ \vdots \\ \\ d_{nz} \end{pmatrix} .
$$

Note that the values a_1 and c_{nz} are not used. Also, since W_1^2 and W_{nz}^2 are zero, the values of c_{nzm} and a_2 are not used. The vectors a, b, c, and d correspond to the input arrays `tmpa`, `tmpb`, `tmpc`, and `tmprhs` in `invert_tridiagonal.f90`, respectively. Once W^2 is found using `invert_tridiagonal.f90`, Θ^2 and ρ^2 are integrated explicitly.

References

Durran, D. and J. Klemp, 1983: A compressible model for the simulation of moist mountain waves. *Monthly Weather Review*, **111 (12)**, 2341–2361.

Klemp, J. B., W. C. Skamarock, and J. Dudhia, 2007: Conservative split-explicit time integration methods for the compressible nonhydrostatic equations. *Monthly Weather Review*.

Skamarock, W. and J. Klemp, 1992: The stability of time-split numerical methods for the hydrostatic and the nonhydrostatic elastic equations. *Monthly Weather Review*, **120 (9)**, 2109–2127.