# Das Atmosphärische Modell (DAM)

## Governing Equations

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The prognostic variables in Das Atmosphärische Modell (DAM) are the momenta, virtual potential temperature, density (of moist air), and the mass fractions (mass of component per mass of moist air) for six classes of water. To simplify the following equations, however, we will write down the governing equations for only three classes of water: vapor, liquid, and solid. This allows the following discussion to be generally applicable to any microphysical scheme. It is straightforward to disaggregate the governing equations for liquid and solid into equations for any number of water classes (e.g., cloud, ice, rain, show, graupel, etc.) The mass fractions for water vapor, liquid, and solid are denoted by $q_v$, $q_l$, and $q_s$, respectively. The mass fraction of dry air is, therefore, $q_a = 1 - q_v - q_l - q_s$.

The velocity of dry air is denoted by $\vec{u}$ and the velocities of the water components are denoted by $\vec{v}_v$, $\vec{v}_l$, and $\vec{v}_s$. Each of these three velocities is the sum of the dry air velocity, diffusion velocity relative to dry air, and the free-fall terminal velocity relative to dry air. We will define the fluxes $\vec{d}_v$, $\vec{d}_l$, and $\vec{d}_s$ as the mass fluxes of the three water classes in the reference frame of the dry air,

$$\begin{aligned} \vec{d}_j &= q_j \rho (\vec{v}_j - \vec{u}) \\ \vec{d}_l &= q_l \rho (\vec{v}_l - \vec{u}) \\ \vec{d}_s &= q_s \rho (\vec{v}_s - \vec{u}) \,. \end{aligned}$$

Although it is understood that the fluxes $\vec{d}$ contain the flux associated with free fall, we will nonetheless refer to them as diffusion fluxes.

Although the true specific momenta of the water components are hte $\vec{v}$ velocities and the true specific kinetic energies are the $v^2/2$, the momenta and kinetic energies associated with the diffusion fluxes will be neglected. With this approximation, liquid water advects momentum density $q_l \rho \vec{u}$ with velocity $\vec{v}_l$. In addition, the kinetic energy density of a moist parcel is $\rho u^2/2$ regardless of its water content. These approximations are self-consistent, contain only negligible differences with equations that keep the diffusion momenta, and allow a huge simplification of the governing equations.

In writing and manipulating the equations of motion, it will prove advantageous to use a four-vector notation, whereby Greek indices range over $\{0, 1, 2, 3\}$, which correspond to $\{t, x, y, z\}$. Since velocities are well within the classical limit, the zeroth component of any velocity four-vector is always unity. For example,

$$\begin{aligned} u^0 &= 1 \\ u^1 &= u \\ u^2 &= v \\ u^3 &= w \,. \end{aligned}$$

The four-vector notation has the advantage of allowing the tendency and flux divergence of any specific quantity,

$$\frac{\partial}{\partial t}(\rho X) + \vec{\nabla} \cdot (\rho \vec{u} X) \,,$$

to be written in a more compact form,

$$\partial_\alpha (\rho u^\alpha X) \,.$$

Transformations from one class of water to another are tracked by evaporation and melting, $e$ and $m$, which have units of a kg/m$^3$/s.

The total pressure, $p$, is the sum of the partial pressures of dry air and water vapor, $p_a$ and $p_v$. The partial pressures are given by the ideal gas law,

$$\begin{aligned} p_a &= q_a R_a \rho T \\ p_v &= q_v R_v \rho T \,, \end{aligned}$$

where $R_a$ is the specific gas constant for dry air, and $R_v$ is the specific gas constant for water vapor. The four components of moist air – dry air, vapor, liquid, and solid – are treated as having constant specific heat capacities at constant volume: $c_{va}$, $c_{vv}$, $c_{vl}$, and $c_{vs}$, respectively. The constant-pressure specific heat capacities of dry air, vapor, liquid, and solid are

$$c_{pa} = c_{va} + R_a \qquad c_{pv} = c_{vv} + R_v \qquad c_{pl} = c_{vl} \qquad c_{ps} = c_{vs}\,,$$

The gas constant of moist air, $R_m$, and the constant-volume specific heat capacity of moist air, $c_{vm}$, are given by

$$\begin{aligned} c_{vm} &= q_a c_{va} + q_v c_{vv} + q_l c_{vl} + q_s c_{vs} \\ R_m &= q_a R_a + q_v R_v\,. \end{aligned}$$

The constant-pressure specific heat capacity of moist air is, of course, $c_{pm} = c_{vm} + R_m$.

The total specific energies (internal plus kinetic plus potential) of each of the components are

$$\begin{aligned} E_a^{\mathrm{tot}} &= c_{va}(T - T_{\mathrm{trip}}) + \frac{1}{2}u^2 + \phi \\ E_v^{\mathrm{tot}} &= c_{vv}(T - T_{\mathrm{trip}}) + \frac{1}{2}u^2 + \phi + E_{0v} \\ E_l^{\mathrm{tot}} &= c_{vl}(T - T_{\mathrm{trip}}) + \frac{1}{2}u^2 + \phi \\ E_s^{\mathrm{tot}} &= c_{vs}(T - T_{\mathrm{trip}}) + \frac{1}{2}u^2 + \phi - E_{0s}\,. \end{aligned}$$

Here, $\phi = gz$ is the specific gravitational potential energy, $T_{\mathrm{trip}} = 273.16$ K is the triple-point temperature, $E_{0v}$ is the specific internal energy of water vapor at the triple point, and $-E_{0s}$ is the specific internal energy of ice at the triple point.

The quantity $Q$, which has units of a power density, represents the heating sources from electromagnetic radiation and thermal conduction. The rank-two tensor $\overleftrightarrow{\tau}$ is the stress tensor. The friction force and dissipation relate to the stress tensor by

$$\begin{aligned} \vec{\mathcal{F}} &= -\vec{\nabla}\cdot\overleftrightarrow{\tau} \\ \mathcal{D} &= -\tau_{ij}\nabla_i u_j\,. \end{aligned}$$

With these definitions, the governing equations may be written as

$$\begin{aligned} \partial_\alpha\left(q_a\rho u^\alpha\right) &= 0 \\ \partial_\alpha\left(q_v\rho v_v^\alpha\right) &= e \\ \partial_\alpha\left(q_l\rho v_l^\alpha\right) &= -e + m \\ \partial_\alpha\left(q_s\rho v_s^\alpha\right) &= -m \\ \partial_\alpha\left(q_a\rho u^\alpha\vec{u}\right) + \partial_\alpha\left(q_v\rho v_v^\alpha\vec{u}\right) & \\ + \partial_\alpha\left(q_l\rho v_l^\alpha\vec{u}\right) + \partial_\alpha\left(q_s\rho v_s^\alpha\vec{u}\right) &= \rho\vec{g} - \vec{\nabla}p - \vec{\nabla}\cdot\vec{\tau} \\ \partial_\alpha\left(q_a\rho u^\alpha E_a^{\mathrm{tot}}\right) + \partial_\alpha\left(q_v\rho v_v^\alpha E_v^{\mathrm{tot}}\right) & \\ + \partial_\alpha\left(q_l\rho v_l^\alpha E_l^{\mathrm{tot}}\right) + \partial_\alpha\left(q_s\rho v_s^\alpha E_s^{\mathrm{tot}}\right) &= Q - \vec{\nabla}\cdot(p_a\vec{u}) - \vec{\nabla}\cdot(p_v\vec{v}_v) \\ &\quad - \vec{\nabla}\cdot(\vec{u}\cdot\vec{\tau}) - \vec{\nabla}\cdot\vec{J}\,. \end{aligned}$$

These equations can be rearranged to give

$$\partial_\alpha(\rho u^\alpha) = -\vec{\nabla}\cdot(\vec{d}_v + \vec{d}_l + \vec{d}_s) \tag{1}$$

$$\partial_\alpha(q_v\rho u^\alpha) = e - \vec{\nabla}\cdot\vec{d}_v \tag{2}$$

$$\partial_\alpha(q_l\rho u^\alpha) = -e + m - \vec{\nabla}\cdot\vec{d}_l \tag{3}$$

$$\partial_\alpha(q_s\rho u^\alpha) = -m - \vec{\nabla}\cdot\vec{d}_s \tag{4}$$

$$\partial_\alpha(\rho u^\alpha\vec{u}) = \rho\vec{g} - \vec{\nabla}p - \vec{\nabla}\cdot\vec{\tau} - \partial_i\left((d_v^i + d_l^i + d_s^i)\vec{u}\right) \tag{5}$$

$$\begin{aligned} \partial_\alpha(\rho u^\alpha E^{\mathrm{tot}}) &= Q - \vec{\nabla}\cdot(p\vec{u}) - \vec{\nabla}\cdot(\vec{u}\cdot\vec{\tau}) - \vec{\nabla}\cdot\vec{J} \\ &\quad - \vec{\nabla}\cdot\left((E_v^{\mathrm{tot}} + R_vT)\vec{d}_v\right) - \vec{\nabla}\cdot\left(E_l^{\mathrm{tot}}\vec{d}_l\right) - \vec{\nabla}\cdot\left(E_s^{\mathrm{tot}}\vec{d}_s\right)\,, \end{aligned} \tag{6}$$

where $E^{\rm tot} = c_{vm}(T - T_{\rm trip}) + \frac{1}{2}u^2 + \phi + q_v E_{0v} - q_s E_{0s}$.

We can derive from these

$$
\begin{aligned}
\partial_\alpha \left(R_m \rho u^\alpha\right) &= R_v(e - \vec{\nabla}\cdot\vec{d}_v) \\
\partial_\alpha \left(c_{vm} \rho u^\alpha\right) &= c_{vv}(e - \vec{\nabla}\cdot\vec{d}_v) + c_{vl}(-e + m - \vec{\nabla}\cdot\vec{d}_l) + c_{vs}(-m - \vec{\nabla}\cdot\vec{d}_s) \\
\partial_\alpha \left(c_{pm} \rho u^\alpha\right) &= c_{pv}(e - \vec{\nabla}\cdot\vec{d}_v) + c_{vl}(-e + m - \vec{\nabla}\cdot\vec{d}_l) + c_{vs}(-m - \vec{\nabla}\cdot\vec{d}_s) \\
\partial_\alpha \left(\rho u^\alpha \frac{1}{2} u^2\right) &= -\vec{\nabla}\cdot\left(\frac{1}{2}u^2(\vec{d}_v + \vec{d}_l + \vec{d}_s)\right) + \rho\vec{g}\cdot\vec{u} - \vec{u}\cdot\vec{\nabla}p + \vec{u}\cdot\vec{\mathcal{F}} \\
\partial_\alpha(\rho\phi u^\alpha) &= -\phi\vec{\nabla}\cdot(\vec{d}_v + \vec{d}_l + \vec{d}_s) - \rho\vec{g}\cdot\vec{u} \\
\partial_\alpha(q_v E_{0v}\rho u^\alpha) &= \left(e - \vec{\nabla}\cdot\vec{d}_v\right) E_{0v} \\
\partial_\alpha(q_s E_{0s}\rho u^\alpha) &= \left(-m - \vec{\nabla}\cdot\vec{d}_s\right) E_{0s} \\
\partial_\alpha(\rho T u^\alpha) &= \frac{1}{c_{vm}}\left\{ Q - p\vec{\nabla}\cdot\vec{u} + \mathcal{D} - c_{vv}\vec{d}_v\cdot\vec{\nabla}T - \vec{\nabla}\cdot\left(R_v T\vec{d}_v\right) - c_{vl}\vec{d}_l\cdot\vec{\nabla}T - c_{vs}\vec{d}_s\cdot\vec{\nabla}T \right. \\
&\quad \left. + \vec{g}\cdot\left(\vec{d}_v + \vec{d}_l + \vec{d}_s\right) - \vec{\nabla}\cdot\vec{d}_{\rm dse} - L_e e - L_m m + R_v T e - c_{vm}T\vec{\nabla}\cdot\left(\vec{d}_v + \vec{d}_l + \vec{d}_s\right)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\vec{\mathcal{F}} &= -\vec{\nabla}\cdot\overleftrightarrow{\tau} \\
\mathcal{D} &= -\overleftrightarrow{\tau} : \vec{\nabla}\vec{u}\,.
\end{aligned}
$$

Given the governing equations, we want to define a thermodynamic variable that is conserved for transformations that are adiabatic, non-dissipative, and have neither phase changes nor diffusion of water. A quantity that satisfies these conditions is the virtual potential temperature,

$$
\theta_v = \frac{R_m}{R_a} T \left(\frac{p_0}{p}\right)^{R_m/c_{pm}} .
$$

We call this the "virtual potential temperature" because it is equal to the potential temperature for a parcel with fixed gas constant and heat capacity ($R_m$ and $c_{pm}$) times the factor $R_m/R_a$, which accounts for virtual effects on density. In particular, if a moist parcel has a temperature $T$, a dry parcel at the same pressure must have the temperature $(R_m/R_a)T$ in order to have the same density. To see this, not that, if parcels 1 and 2 have equal pressures, then $p_1 = p_2$. The ideal gas law then gives $R_1\rho_1 T_1 = R_2\rho_2 T_2$. If the densities are also equal, then $R_1 T_1 = R_2 T_2$, or $T_1 = (R_2/R_1)T_2$.

We can also write the expression for $\theta_v$ as

$$
\rho\theta_v = \frac{p_0}{R_a}\left(\frac{p}{p_0}\right)^{1/\gamma_m} ,
$$

where $\gamma = c_{pm}/c_{vm}$. Density and temperature are related to $\theta_v$ via the following relations:

$$
\begin{aligned}
p &= p_0\left(\frac{R_a\rho\theta_v}{p_0}\right)^{\gamma_m} \\
T &= \frac{R_a}{R_m}\theta_v\left(\frac{R_a\rho\theta_v}{p_0}\right)^{\gamma_m - 1} .
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\partial_\alpha\left[(p/p_0)^{1/\gamma}\right] &= \partial_\alpha\left[e^{\frac{1}{\gamma}\log(p/p_0)}\right] \\
&= e^{\frac{1}{\gamma}\log(p/p_0)}\left[-\frac{1}{\gamma^2}\log(p/p_0)\,\partial_\alpha\gamma + \frac{1}{\gamma(p/p_0)}\partial_\alpha(p/p_0)\right] \\
&= \frac{1}{\gamma}(p/p_0)^{\frac{1}{\gamma}-1}\partial_\alpha(p/p_0) - \frac{1}{\gamma^2}(p/p_0)^{\frac{1}{\gamma}}\log(p/p_0)\,\partial_\alpha\gamma\,,
\end{aligned}
$$

we can write

$$
\begin{aligned}
\partial_\alpha\left(\rho\theta_v u^\alpha\right) &= \frac{p_0}{R_a}\partial_\alpha\left((p/p_0)^{1/\gamma}u^\alpha\right) \\
&= \frac{p_0}{R_a}\left(\frac{p}{p_0}\right)^{1/\gamma}\left[\vec{\nabla}\cdot\vec{u}+\frac{1}{\gamma}p^{-1}u^\alpha\partial_\alpha p-\frac{1}{\gamma^2}\log(p/p_0)u^\alpha\partial_\alpha\gamma\right] \\
&= \frac{p_0}{R_a}\left(\frac{p}{p_0}\right)^{1/\gamma}\left[\frac{R_m}{c_{pm}}\vec{\nabla}\cdot\vec{u}+\frac{T}{\gamma p}\partial_\alpha(R_m\rho u^\alpha)+\frac{R_m}{\gamma p}\partial_\alpha(T\rho u^\alpha)\right. \\
&\quad \left.-\frac{R_mT}{\gamma p}\partial_\alpha(\rho u^\alpha)-\frac{1}{\gamma c_{pm}\rho}\log(p/p_0)\partial_\alpha\left(c_{pm}\rho u^\alpha\right)+\frac{1}{c_{pm}\rho}\log(p/p_0)\partial_\alpha\left(c_{vm}\rho u^\alpha\right)\right] \\
&= \frac{1}{R_a}\left(\frac{p_0}{p}\right)^{R_m/c_{pm}}\left\{R_vT(e-\vec{\nabla}\cdot\vec{d}_v)\right. \\
&\quad +\left(\frac{R_m}{c_{pm}}\right)^2T\log\left(\frac{p}{p_0}\right)\left[(c_{vv}-c_{vl})e+(c_{vl}-c_{vs})m\right. \\
&\quad \left.-\vec{\nabla}\cdot\left(c_{vv}\vec{d}_v+c_{vl}\vec{d}_l+c_{vs}\vec{d}_s\right)-\frac{c_{vm}}{R_m}R_v(e-\vec{\nabla}\cdot\vec{d}_v)\right] \\
&\quad +\frac{R_m}{c_{pm}}\left[Q+\mathcal{D}-\left(c_{pv}\vec{d}_v+c_{vl}\vec{d}_l+c_{vs}\vec{d}_s\right)\cdot\vec{\nabla}T\right. \\
&\quad \left.\left.+\vec{g}\cdot(\vec{d}_v+\vec{d}_l+\vec{d}_s)-\vec{\nabla}\cdot\vec{J}-L_e e-L_m m\right]\right\}.
\end{aligned}
$$

Therefore, we see that the sources and sinks for $\theta_v$ are zero so long as $Q$, $\vec{J}$, $\mathcal{D}$, $e$, $m$, $\vec{d}_v$, $\vec{d}_l$, and $\vec{d}_s$ are zero.

In summary, the governing equations used by DAM are

$$
\begin{aligned}
\partial_\alpha(\rho u^\alpha) &= -\vec{\nabla}\cdot(\vec{d}_v+\vec{d}_l+\vec{d}_s) \\
\partial_\alpha(q_v\rho u^\alpha) &= e-\vec{\nabla}\cdot\vec{d}_v \\
\partial_\alpha(q_l\rho u^\alpha) &= -e+m-\vec{\nabla}\cdot\vec{d}_l \\
\partial_\alpha(q_s\rho u^\alpha) &= -m-\vec{\nabla}\cdot\vec{d}_s \\
\partial_\alpha(\rho u^\alpha\vec{u}) &= \rho\vec{g}-\vec{\nabla}p-\vec{\nabla}\cdot\vec{\tau}-\partial_i\left((d_v^i+d_l^i+d_s^i)\vec{u}\right) \\
\partial_\alpha\left(\rho\theta_v u^\alpha\right) &= \frac{1}{R_a}\left(\frac{p_0}{p}\right)^{R_m/c_{pm}}\left\{R_vT(e-\vec{\nabla}\cdot\vec{d}_v)\right. \\
&\quad +\left(\frac{R_m}{c_{pm}}\right)^2T\log\left(\frac{p}{p_0}\right)\left[(c_{vv}-c_{vl})e+(c_{vl}-c_{vs})m\right. \\
&\quad \left.-\vec{\nabla}\cdot\left(c_{vv}\vec{d}_v+c_{vl}\vec{d}_l+c_{vs}\vec{d}_s\right)-\frac{c_{vm}}{R_m}R_v(e-\vec{\nabla}\cdot\vec{d}_v)\right] \\
&\quad +\frac{R_m}{c_{pm}}\left[Q+\mathcal{D}-\left(c_{pv}\vec{d}_v+c_{vl}\vec{d}_l+c_{vs}\vec{d}_s\right)\cdot\vec{\nabla}T\right. \\
&\quad \left.\left.+\vec{g}\cdot(\vec{d}_v+\vec{d}_l+\vec{d}_s)-\vec{\nabla}\cdot\vec{J}-L_e e-L_m m\right]\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
T &= \frac{R_a}{R_m}\theta_v\left(\frac{R_a\rho\theta_v}{p_0}\right)^{R_m/c_{vm}} \\
p &= q_aR_a\rho T+q_vR_v\rho T\,.
\end{aligned}
$$